Cyclic cohomology of Hopf algebras of transverse symmetries in codimension 1

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Abstract

We develop intrinsic tools for computing the periodic Hopf cyclic cohomology of Hopf algebras related to transverse symmetry in codimension 1. Besides the Hopf algebra found by Connes and the first author in their work on the local index formula for transversely hypoelliptic operators on foliations, this family includes its 'Schwarzian' quotient, on which the Rankin-Cohen universal deformation formula is based, the extended Connes-Kreimer Hopf algebra related to renormalization of divergences in QFT, as well as a series of cyclic coverings of these Hopf algebras, motivated by the treatment of transverse symmetry for nonorientable foliations.

The method for calculating their Hopf cyclic cohomology is based on two computational devices, which work in tandem and complement each other: one is a spectral sequence for bicrossed product Hopf algebras and the other a Cartan-type homotopy formula in Hopf cyclic cohomology.

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Introduction

Connes' general definition of cyclic cohomology as a derived functor over the cyclic category [3] allows the flexibility of applying it well beyond the realm of algebras, for which it was originally formulated [2, 4] as a noncommutative de Rham theory. In particular, the process of computing the local index formula for transversely hypoelliptic operators on foliations has led to a cyclic cohomology apparatus specific to Hopf algebras [7]. Unfolding the algebraic meaning of the iterated commutator operations involved in the local index formula [6] revealed the existence of a 'universal' Hopf algebra $\mathcal{H}_n$, that plays for the normal bundle to a codimension n foliation a role analogous to that of the affine group of the frame bundle to a manifold of dimension n. Moreover, the calculation itself turned out to be governed by the cyclic cohomology of a specific type of cyclic module structure associated to Hopf algebras. The solution provided in [7] to the transverse index problem was then based on proving the existence of an explicit isomorphism between the periodic cyclic cohomology of the cyclic module associated to the Hopf algebra $\mathcal{H}_n$ and the Gelfand-Fuks cohomology of the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$, thus making the link with the 'classical' theory of characteristic classes of foliations.

The proof given in [7] to the above isomorphism was constructive and shed valuable light on how to perform explicit computations in the realm of Hopf cyclic cohomology, but from a technical viewpoint the end result was still imported from Lie algebra cohomology. It thus remained highly desirable to develop intrinsic methods, apt to directly compute the cyclic cohomology of Hopf algebras such as $\mathcal{H}_n$. Furthermore, one could reasonably expect these methods to require considerably less computational effort in the codimension 1 case.

In this paper we present a direct method for computing the periodic Hopf cyclic cohomology of a whole class of Hopf algebras related to transverse symmetry in codimension 1. Besides $\mathcal{H}_1$, this class contains a series of Hopf algebras whose periodic Hopf cyclic cohomology was not previously calculated. This family includes the 'Schwarzian' quotient $\mathcal{H}_{1s}$ of $\mathcal{H}_1$, which plays a pivotal role in the universal deformation formula [11] based on the Rankin-Cohen brackets on modular forms, as well as the extended version of the Connes-Kreimer Hopf algebra [5] related to the renormalization of divergences in QFT. It also contains a series of 'cyclic coverings' of these Hopf

algebras, obtained as coextensions of them by group rings of cyclic groups.

The cyclic coverings of $\mathcal{H}_1$ are introduced in §1, starting from the observation that the process of extension of the Hopf algebra symmetries to the case of transversely nonorientable foliations leads naturally to the construction of a certain 'double cover' $\mathcal{H}_1^{\dagger|2}$ (see §1.1). A similarly defined 'infinite cyclic cover' $\mathcal{H}_1^{\dagger}$ (see §1.2) is then shown to act effectively on Hecke algebras related to modular forms, of the type introduced in [10] (see §1.3).

The tool kit we employ in order to calculate the periodic cyclic cohomology of the above family of Hopf algebras consists of two computational devices, which fortunately, albeit rather fortuitously, complement each other. The first is the construction of a bicocyclic module for a class of bicrossed product Hopf algebras of the type mentioned above, that allows the computation of their Hopf cyclic cohomology (see Theorems 2.11 and 2.18) by means of an Eilenberg-Zilber theorem for byparacyclic modules, cf. [14, 16]. The second is a Cartan homotopy formula for the Hopf cyclic cohomology of coalgebras with coefficients in SAYD modules (Proposition 3.7). Neither one of these tools would suffice by itself to achieve the desired goal. However, when applied in tandem they allow the calculation of the periodic Hopf cyclic cohomology of $\mathcal{H}_1$ (Theorem 4.4) and of the Schwarzian factor Hopf algebra $\mathcal{H}_{1s}$ (Theorem 4.5) without relying on Gelfand-Fuks cohomology. Moreover, they also allow us to compute the periodic Hopf cyclic cohomology of the cyclic cover $\mathcal{H}_1^{\dagger}$ (with coefficients in various modular pairs in involution, cf. Theorem 4.8), as well as to calculate the periodic Hopf cyclic cohomology of the extended Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}$ (Theorem 4.10) and of its own cyclic covers (Theorem 4.11).

Contents

1 Coverings of $\mathcal{H}_1$ by cyclic groups

After briefly reviewing some basic facts about the Hopf algebra $\mathcal{H}_1$ (cf. [7]), we shall explain how the consideration of the transversely nonorientable case leads naturally to the introduction of a 'double cover' $\mathcal{H}_1^{\dagger|2}$ and, more generally, of an infinite cyclic cover $\mathcal{H}_1^\dagger$. The latter will be shown to act effectively as 'symmetries' of modular Hecke algebras (comp. [10]).

1.1 $\mathcal{H}_1$ and its periodic classes

Given a discrete subgroup of orientation preserving diffeomorphisms $\Gamma \subset \mathrm{Diff}^+(S^1)$, let F^+S^1 denote the bundle of positively oriented frames over S^1, on which the diffeomorphisms $\varphi \in \Gamma$ act in the obvious natural manner:

$$\widetilde{\varphi}(x,y) = (\varphi(x),\, \varphi'(x)\cdot y)\,, \qquad (x,y) \in F^+S^1 \simeq (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^+\,.$$

One then forms the crossed product algebra

$$\mathcal{A}_\Gamma^+ \,=\, C_c^\infty(F^+S^1) \rtimes \Gamma\,,$$

which can be viewed as the linear span of the set of monomials $\{fU_\varphi\,;\, f \in C_c^\infty(F^+S^1),\, \varphi \in \Gamma\}$, endowed with the product

$$fU_\varphi \cdot gU_\psi = f\,(g \circ \varphi^{-1})\, U_{\varphi\psi}\,.$$

The algebra $\mathcal{A}_\Gamma^+$ typifies the 'coordinates' of the 'space of leaves' of a *transversely orientable* codimension 1 foliation.

The trivial connection on the frame bundle $F^+S^1 \to S^1$ is implemented by the vector fields

$$Y \,=\, y\,\frac{\partial}{\partial y} \qquad \text{and} \qquad X \,=\, y\,\frac{\partial}{\partial x}\,, \tag{1.1}$$

the first being the generator of the vertical sub-bundle and the second the generator of the horizontal sub-bundle. These two basic vector fields become linear operators on $\mathcal{A}_\Gamma^+$, acting as

$$Y(f\,U_\varphi) = Y(f)\,U_\varphi\,, \qquad X(f\,U_\varphi) = X(f)\,U_\varphi\,. \tag{1.2}$$

The prolongation of Y acts as derivation

$$Y(ab) = Y(a)\,b + a\,Y(b)\,, \qquad a, b \in \mathcal{A}_\Gamma^+\,, \tag{1.3}$$

because Y is invariant under the action of $\mathrm{Diff}^+(S^1)$ on F^+S^1. On the other hand,

$$X(f\circ\varphi)(x,y) = X(f)(\varphi(x),\varphi'(x)\cdot y) + y\frac{\varphi''(x)}{\varphi'(x)}Y(f)(\varphi(x),\varphi'(x)\cdot y) \tag{1.4}$$

which is tantamount with the following 'Leibniz rule' for its prolongation:

$$X(ab) = X(a)\,b \,+\, a\,X(b) + \delta_1(a)\,Y(b)\,, \tag{1.5}$$

where

$$\delta_1(f\,U_{\varphi^{-1}}) \;=\; y\,\frac{d}{dx}\left(\log\frac{d\varphi}{dx}\right)\,fU_{\varphi^{-1}}\,. \tag{1.6}$$

In turn, the operator δ_1 is itself a derivation,

$$\delta_1(ab) = \delta_1(a)\,b + a\,\delta_1(b)\,, \tag{1.7}$$

and its successive commutators with X produces new operators

$$\delta_n(f\,U_{\varphi^{-1}}) \;=\; y^n\,\frac{d^n}{dx^n}\left(\log\frac{d\varphi}{dx}\right)\,fU_{\varphi^{-1}}\,, \qquad \forall\, n \geq 1\,, \tag{1.8}$$

that satisfy progressively more complicated Leibniz rules.

The formulae (1.3), (1.5), (1.7) in conjunction with (1.2), (1.6), (1.8) express precisely the fact that they define a Hopf action of the Hopf algebra $\mathcal{H}_1$ on $\mathcal{A}_\Gamma^+$. As an algebra $\mathcal{H}_1$ is generated by X, Y, δ_k, $k \in \mathbb{N}$, subject to the relations

$$[Y, X] = X, \qquad [Y,\delta_k] = k\delta_k, \qquad [X,\delta_k] = \delta_{k+1}, \qquad [\delta_j,\delta_k] = 0. \tag{1.9}$$

Its coalgebra structure is uniquely determined by

$$\Delta(Y) = Y \otimes 1 + 1 \otimes Y, \tag{1.10}$$
$$\Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \tag{1.11}$$
$$\Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1, \tag{1.12}$$

$$\epsilon(X) = \epsilon(Y) = \epsilon(\delta_k) = 0, \tag{1.13}$$

together with the requirement that $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$ is multiplicative. The antipode is the unique antihomomorphism $S : \mathcal{H}_1 \to \mathcal{H}_1$ defined on generators by

$$S(X) = -X + \delta_1 Y, \quad S(Y) = -Y \quad \text{and} \quad S(\delta_1) = -\delta_1 \,. \tag{1.14}$$

Furthermore, the above action admits an invariant trace. Indeed, the volume form $\dfrac{dx \wedge dy}{y^2}$ on F^+S^1 is invariant under $\mathrm{Diff}^+(S^1)$ and thus gives rise to a trace $\tau^+ : \mathcal{A}_\Gamma^+ \to \mathbb{C}$,

$$\tau^+(fU_\varphi) = \begin{cases} \displaystyle\int_{F^+S^1} f(x,y)\, \frac{dx \wedge dy}{y^2}\,, & \text{if } \varphi = 1\,, \\ 0\,, & \text{if } \varphi \neq 1\,. \end{cases} \tag{1.15}$$

This trace satisfies the invariance property

$$\tau^+(h(a)) = \delta(h)\, \tau^+(a)\,, \qquad \forall \quad h \in \mathcal{H}_1\,, \tag{1.16}$$

where $\delta \in \mathcal{H}_1^*$ is the 'modular' character, defined on generators by

$$\delta(Y) = 1, \quad \delta(X) = 0, \quad \delta(\delta_n) = 0\,. \tag{1.17}$$

To prove the identity (1.16), it suffices to note that it holds for X and Y by an elementary integration by parts argument, and is trivially fulfilled by δ_1 since $\delta_1(\mathrm{Id}) = 0$.

For a quick exposition of the basic notions and notation regarding Hopf cyclic cohomology, the reader is referred to [8] (see also §2.1 below). In particular,

a *modular pair* for a Hopf algebra $\mathcal{H}$ is a pair (σ, δ), where σ is a group-like element of $\mathcal{H}$ and $\delta : \mathcal{H} \to \mathbb{C}$ is an algebra map such that $\delta(\sigma) = 1$. It is called *modular pair in involution*, to be abbreviated henceforth *MPI*, iff denoting $S_\delta = \delta * S$, the following condition is satisfied:

$$S_\delta^2(h) = \sigma h \sigma^{-1}, \quad \forall\, h \in \mathcal{H}. \tag{1.18}$$

The pair $(\delta, 1)$ forms such an MPI, so that the corresponding Hopf cyclic cohomology $HC^*_{(\delta,1)}(\mathcal{H}_1)$ is well-defined. Moreover, the assignment

$$\chi^n_{\tau^+}(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \tau^+(a^0\, h^1(a^1) \ldots h^n(a^n)), \quad h^i \in \mathcal{H}_1,\ a^j \in \mathcal{A}_\Gamma^+$$

induces a characteristic homomorphism in cyclic cohomology (cf. [7, 8])

$$\chi^*_{\tau^+} : HC^*_{(\delta,1)}(\mathcal{H}_1) \to HC^*(\mathcal{A}_\Gamma^+). \tag{1.19}$$

As is well-known, in codimension 1 there are just two linearly independent geometric characteristic classes of foliations: the 0-dimensional Pontryagin class $p_0 = 1$ and the Godbillon-Vey class. In the framework of cyclic cohomology, the first becomes the *transverse fundamental class* and is represented by the cyclic 2-cocycle $TF_\Gamma \in ZC^2(\mathcal{A}_\Gamma^+)$,

$$TF_\Gamma(f^0 U_{\varphi_0^{-1}}, f^1 U_{\varphi_1^{-1}}, f^2 U_{\varphi_2^{-1}}) = \begin{cases} \displaystyle\int_{F^+S^1} f^0 \widetilde{\varphi}_0^*(df^1) \widetilde{\varphi}_0^* \widetilde{\varphi}_1^*(df^2), & \varphi_2\varphi_1\varphi_0 = \mathrm{Id} \\ 0, & \varphi_2\varphi_1\varphi_0 \neq \mathrm{Id}, \end{cases} \tag{1.20}$$

and the second by the cyclic 1-cocycle

$$GV_\Gamma(f^0 U_{\varphi_0^{-1}}, f^1 U_{\varphi_1^{-1}}) = \begin{cases} \displaystyle\int_{F^+S^1} f^0\, \widetilde{\varphi}_0^*(f^1)\, y \frac{\varphi_0''(x)}{\varphi_0'(x)} \frac{dx \wedge dy}{y^2}, & \varphi_1\varphi_0 = \mathrm{Id} \\ 0, & \varphi_1\varphi_0 \neq \mathrm{Id}, \end{cases} \tag{1.21}$$

By an elementary calculation (see [12, Prop. 18]), the first cocycle can be expressed in the form

$$\begin{aligned} &TF_\Gamma(f^0 U_{\varphi_0^{-1}}, f^1 U_{\varphi_1^{-1}}, f^2 U_{\varphi_2^{-1}}) = \tau^+(f^0 U_{\varphi_0^{-1}}\, X(f^1 U_{\varphi_1^{-1}})\, Y(f^2 U_{\varphi_2^{-1}}) \\ -\ &\tau^+(f^0 U_{\varphi_0^{-1}}\, Y(f^1 U_{\varphi_1^{-1}})\, X(f^2 U_{\varphi_2^{-1}}) - \tau^+(f^0 U_{\varphi_0^{-1}}\, \delta_1 Y(f^1 U_{\varphi_1^{-1}})\, Y(f^2 U_{\varphi_2^{-1}}) \end{aligned}$$

which can be recognized as the image by the characteristic map (1.19) of the Hopf cyclic cocycle $TF \in ZC^2_{(\delta,1)}(\mathcal{H}_1)$

$$TF \,=\, X \otimes Y - Y \otimes X - \delta_1 \, Y \otimes Y \,\in\, \mathcal{H}_1 \otimes \mathcal{H}_1 \,. \tag{1.22}$$

Using the invariance property (1.16) for $h = \delta_1$, the Godbillon-Vey cocycle (1.21) can be rewritten as

$$\begin{aligned} GV_\Gamma(f^0 U_{\varphi_0^{-1}}, f^1 U_{\varphi_1^{-1}}) &= \tau(\delta_1(f^0 U_{\varphi_0^{-1}})\, f^1 U_{\varphi_1^{-1}}) = -\tau(f^0 U_{\varphi_0^{-1}}\, \delta_1(f^1 U_{\varphi_1^{-1}})) \\ &= -\chi^1_\tau(\delta_1)(f^0 U_{\varphi_0^{-1}}, f^1 U_{\varphi_1^{-1}})\,, \end{aligned}$$

which is obviously the image by the characteristic map of the Hopf cyclic cocycle

$$GV \,=\, -\delta_1 \,\in\, ZC^1_{(\delta,1)}(\mathcal{H}_1)\,.$$

1.2 $\mathcal{H}_1^\dagger$ and its periodic classes

We now proceed to upgrade the above constructions in order to cover the case of non-orientable codimension 1 foliations. First of all, this entails replacing the positive frame bundle F^+S^1 by the quotient $FS^1/\mathbb{Z}_2$ of the full frame bundle, where the action of $\mathbb{Z}_2$ is given by the reflection $(x, y) \mapsto (x, -y)$, and allowing Γ to be an arbitrary subgroup of the full diffeomorphism group $\mathrm{Diff}(S^1)$. The corresponding crossed product algebra is then

$$\mathcal{A}_\Gamma \,=\, C_c^\infty(FS^1/\mathbb{Z}_2) \rtimes \Gamma\,,$$

where $C_c^\infty(FS^1/\mathbb{Z}_2)$ is identified with the space of even functions

$$C_c^\infty(FS^1)^{\mathrm{ev}} \,=\, \{f \in C_c^\infty(FS^1)\,; \quad f(x,y) = f(x,-y)\,, \quad \forall\,(x,y) \in S^1 \times \mathbb{R}\}\,.$$

The trivial connection is now implemented by the vector fields $\{Y, X\}$, where Y is the same as in (1.1) while the expression of X needs an obvious adjustment:

$$Y \,=\, y\,\frac{\partial}{\partial y} \qquad \text{and} \qquad X \,=\, |y|\,\frac{\partial}{\partial x}\,. \tag{1.23}$$

As a consequence, (1.4) becomes

$$\begin{aligned} X(f \circ \varphi)(x,y) &= \operatorname{sign} \varphi'(x) \cdot X(f)(\varphi(x), \varphi'(x)\cdot y) \\ &+ \ |y| \frac{\varphi''(x)}{\varphi'(x)} Y(f)(\varphi(x), \varphi'(x)\cdot y)\,, \end{aligned} \tag{1.24}$$

which implies the Leibniz rule

$$X(ab) = X(a)\,b \,+\, \sigma(a)\,X(b) + \delta_1(a)\,Y(b)\,, \tag{1.25}$$

where

$$\begin{aligned} \sigma(f\,U_{\varphi^{-1}}) &= \operatorname{sign}\varphi'(x)\cdot f\,U_{\varphi^{-1}}\,, && (1.26)\\ \text{and} \qquad \delta_1(f\,U_{\varphi^{-1}}) &= |y|\,\frac{d}{dx}\left(\log\frac{d\varphi}{dx}\right) fU_{\varphi^{-1}}\,. && (1.27)\end{aligned}$$

The operator δ_1 also becomes an σ-derivation,

$$\delta_1(ab) = \delta_1(a)\,b + \,\sigma(a)\,\delta_1(b)\,, \tag{1.28}$$

while σ acts as automorphism

$$\sigma(ab) \,=\, \sigma(a)\,\sigma(b)\,. \tag{1.29}$$

The iterated commutators of δ_1 with X give rise to the operators

$$\delta_n(f\,U_{\varphi^{-1}}) \,=\, |y|^n\,\frac{d^n}{dx^n}\left(\log\frac{d\varphi}{dx}\right) fU_{\varphi^{-1}}\,, \qquad \forall\, n\geq 1\,. \tag{1.30}$$

The formulae (1.3), (1.25), (1.28) and (1.29) lead to the definition of a 'central extension' of $\mathcal{H}_1$ by the group ring $\mathbb{C}[\mathbb{Z}]\equiv\mathbb{C}[\sigma,\sigma^{-1}]$,

$$\mathcal{H}_1^\dagger \,:=\, \mathcal{H}_1\otimes\mathbb{C}[\sigma,\sigma^{-1}]\,, \tag{1.31}$$

whose presentation as an algebra is given by (1.9) plus the condition that σ is central. The coalgebra structure of $\mathcal{H}_1^\dagger$ is determined by obvious analogues of the formulae (1.10)–(1.13), namely

$$\begin{aligned} &\Delta(Y) = Y\otimes 1 + 1\otimes Y,\\ &\Delta(X) = X\otimes 1 + \sigma\otimes X + \delta_1\otimes Y,\\ &\Delta(\delta_1) = \delta_1\otimes 1 + \sigma\otimes\delta_1,\\ &\Delta(\sigma) = \sigma\otimes\sigma, \quad \Delta(\sigma^{-1}) = \sigma^{-1}\otimes\sigma^{-1} \end{aligned}$$

$$\epsilon(X) = \epsilon(Y) = \epsilon(\delta_k) = 0, \quad \epsilon(\sigma) = 1,$$

while the corresponding antipode is defined by the relations

$$\begin{aligned} S(\sigma) &= \sigma^{-1}, \quad S(X) = \sigma^{-1}(-X + \delta_1 Y), \\ S(Y) &= -Y, \quad S(\delta_1) = -\sigma^{-1}\delta_1 . \end{aligned}$$

For each integer $N \geq 1$ one can add to the presentation of $\mathcal{H}_1^\dagger$ the relation

$$\sigma^N = 1 ,$$

or equivalently replace $\mathbb{C}[\mathbb{Z}]$ by $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$, and preserve the rest of the axioms to define a Hopf algebra $\mathcal{H}_1^{\dagger|N}$. Alternatively, $\mathcal{H}_1^{\dagger|N}$ can be defined as the quotient of $\mathcal{H}_1^\dagger$ by the ideal generated by $\sigma^N - 1$, which is also a coideal.

At this point we note that, with the notation just introduced, the identities (1.2), (1.26), (1.27) and (1.30) actually define a Hopf action of $\mathcal{H}_1^{\dagger|2}$ on $\mathcal{A}_\Gamma$. Furthermore, the obvious counterpart of the trace τ^+, namely

$$\tau(fU_\varphi) = \begin{cases} \displaystyle\int_{FS^1/\mathbb{Z}_2} f(x,y)\, \frac{dx \wedge dy}{y|y|}, & \text{if } \varphi = 1 , \\ 0, & \text{if } \varphi \neq 1 , \end{cases}$$

gives an invariant σ^{-1}-trace on $\mathcal{A}_\Gamma$, which means that it satisfies the identity

$$\tau(a\, b) = \tau(b\, \sigma^{-1}(a)) , \qquad \forall\, a, b \in \mathcal{A}_\Gamma$$

Indeed, the change of variables $(x, |y|) \mapsto (\varphi(x), |\varphi'(x)||y|)$ gives

$$\begin{aligned} \tau(fU_\varphi \cdot gU_{\varphi^{-1}}) &= \int_{FS^1/\mathbb{Z}_2} f(x,y)\, (g \circ \tilde{\varphi}^{-1}(x,y))\, \frac{dx \wedge dy}{y|y|} \\ &= \int_{FS^1/\mathbb{Z}_2} g(x,y)\, (f \circ \tilde{\varphi}(x,y))\, \mathrm{sign}\, \varphi'(x)\, \frac{dx \wedge dy}{y|y|} \\ &= \tau(\sigma(gU_{\varphi^{-1}})\, fU_\varphi) = \tau(gU_{\varphi^{-1}} \cdot \sigma^{-1}(fU_\varphi)) ; \end{aligned}$$

in the last line we used the invariance property

$$\tau(\sigma(a)) = \tau(a) , \qquad \forall \quad a \in \mathcal{A}_\Gamma ,$$

which is an immediate consequence of the fact that $\sigma(\mathrm{Id}) = 1$.

More generally, as in the case of τ^+ one can easily check that

$$\tau(h(a)) \;=\; \delta(h)\,\tau(a)\,, \qquad \forall \quad h \in \mathcal{H}_1\,; \tag{1.32}$$

here the modular character δ is the extension of that defined by (1.17), with the additional condition

$$\delta(\sigma) = 1\,. \tag{1.33}$$

As (δ, σ^{-1}) is obviously an involutive modular pair for $\mathcal{H}_1^\dagger$, one obtains a characteristic homomorphism in cyclic cohomology

$$\chi_\tau^* \;:\; HC^*_{(\delta,\sigma^{-1})}(\mathcal{H}_1^\dagger) \;\to HC^*(\mathcal{A}_\Gamma) \tag{1.34}$$

that factors through $\mathcal{H}_1^{\dagger|2}$, by means of the assignment

$$\chi_\tau^n(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \tau(a^0\, h^1(a^1) \ldots h^n(a^n)), \quad h^i \in \mathcal{H}_1^\dagger,\; a^j \in \mathcal{A}_\Gamma.$$

From the very definitions of the two characteristic maps, it follows quite easily that if $\Gamma \subset \mathrm{Diff}(S^1)$ is a discrete subgroup of diffeomorphisms, $\Gamma^+ = \Gamma \cap \mathrm{Diff}^+(S^1)$, and $\iota_\Gamma : \mathcal{A}^+_{\Gamma^+} \to \mathcal{A}_\Gamma$ denotes the obvious inclusion, then the diagram

$$\begin{tikzcd}
HC^*_{(\delta,\sigma^{-1})}(\mathcal{H}_1^\dagger) \arrow[r, "\pi_1^*"] \arrow[d, "\chi_\tau^*"] & HC^*_{(\delta,1)}(\mathcal{H}_1) \arrow[d, "\chi_{\tau^+}^*"] \\
HC^*(\mathcal{A}_\Gamma) \arrow[r, "\iota_\Gamma^*"] & HC^*(\mathcal{A}^+_{\Gamma^+})
\end{tikzcd}$$

is commutative; here $\pi_1 : \mathcal{H}_1^\dagger \to \mathcal{H}_1$ stands for the natural projection that sends σ to 1 and ${\pi_1}^*$ for the map induced on cohomology.

The two classes given by the cyclic cocycles (1.20) and (1.21) lift naturally to classes in $HC^*(\mathcal{A}_\Gamma)$:

$$TF_\Gamma(f^0 U_{\varphi_0^{-1}}, f^1 U_{\varphi_1^{-1}}, f^2 U_{\varphi_2^{-1}}) = \begin{cases} \displaystyle\int_{FS^1/\mathbb{Z}_2} f^0 \widetilde{\varphi}_0^*(df^1) \widetilde{\varphi}_0^* \widetilde{\varphi}_1^*(df^2), & \varphi_2\varphi_1\varphi_0 = \mathrm{Id} \\ 0\,, & \varphi_2\varphi_1\varphi_0 \neq \mathrm{Id}\,, \end{cases} \tag{1.35}$$

respectively

$$GV_\Gamma(f^0U_{\varphi_0^{-1}}, f^1U_{\varphi_1^{-1}}) = \begin{cases} \displaystyle\int_{FS^1/\mathbb{Z}_2} f^0\, \widetilde{\varphi}_0^*(f^1)\, |y| \frac{\varphi_0''(x)}{\varphi_0'(x)}\, \frac{dx\wedge dy}{y|y|}, & \varphi_1\varphi_0 = \mathrm{Id} \\ 0\,, & \varphi_1\varphi_0 \neq \mathrm{Id}\,, \end{cases} \tag{1.36}$$

As in the orientable case, we can identify them as characteristic images of classes in $HC^*_{(\delta,\sigma^{-1})}(\mathcal{H}_1^\dagger)$.

Proposition 1.1. *The elements*

$$\begin{array}{rcl} GV^\dagger & = & -\sigma^{-1}\delta_1\,, \\ TF^\dagger & = & \sigma^{-1}X\otimes\sigma^{-1}Y - Y\otimes\sigma^{-1}X - \sigma^{-1}\delta_1 Y\otimes\sigma^{-1}Y \end{array}$$

are cyclic cocycles in the Hopf cyclic module associated to the involutive modular pair $(\mathcal{H}_1^\dagger;\delta,\sigma^{-1})$*, and they satisfy the naturality property*

$$\begin{array}{rcl} \chi_\tau^*(GV^\dagger) & = & GV_\Gamma\,, \qquad \pi_1^{2*}(GV^\dagger) = GV\,, \\ \chi_\tau^*(TF^\dagger) & = & TF_\Gamma\,, \qquad \pi_1^{2*}(TF^\dagger) = TF\,. \end{array}$$

Proof. Starting with the Godbillon-Vey cocycle, we note that formula (1.36) is equivalent to

$$GV_\Gamma(f^0U_{\varphi_0^{-1}}, f^1U_{\varphi_1^{-1}}) \;=\; \tau(\delta_1(f^0U_{\varphi_0^{-1}})\, f^1U_{\varphi_1^{-1}})\,,$$

and using the invariance property (1.32), for δ_1 and then for σ, it follows that

$$\begin{array}{l} GV_\Gamma(f^0U_{\varphi_0^{-1}}, f^1U_{\varphi_1^{-1}}) \;=\; -\tau(\sigma(f^0U_{\varphi_0^{-1}})\,\delta_1(f^1U_{\varphi_1^{-1}})) = \\ \quad = -\tau(f^0U_{\varphi_0^{-1}}\,\sigma^{-1}\delta_1(f^0U_{\varphi_0^{-1}})) \;=\; -\chi_\tau(\sigma^{-1}\delta_1)(f^0U_{\varphi_0^{-1}}, f^1U_{\varphi_1^{-1}})\,, \end{array}$$

which expresses it as the image by the characteristic map (1.34) of

$$GV^\dagger \;=\; -\sigma^{-1}\delta_1 \;\in\; C^1_{(\delta,1)}(\mathcal{H}_1)\,.$$

The latter is a Hochschild cocycle, since

$$b(\sigma^{-1}\delta_1) = 1\otimes\sigma^{-1}\delta_1 - (\sigma^{-1}\otimes\sigma^{-1})(\delta_1\otimes 1 + \sigma\otimes\delta_1) + \sigma^{-1}\delta_1\otimes\sigma^{-1} = 0\,,$$

and is also cyclic, because

$$\tau_1(-\sigma^{-1}\delta_1) \,=\, -\tilde{S}(\sigma^{-1}\delta_1))\,\sigma^{-1} = \,-\tilde{S}(\delta_1) \,=\, \sigma^{-1}\delta_1\,.$$

Passing to the transverse fundamental class, we note first that relative to the framing (1.23) of the cotangent bundle $T^*(FS^1/\mathbb{Z}_2)$, for any $f \in C^\infty(FS^1/\mathbb{Z}_2)$,

$$df \,=\, X(f)\,|y|^{-1}dx \,+\, Y(f)\,y^{-1}dy\,,$$

hence for any $\varphi \in \mathrm{Diff}(S^1)$

$$\widetilde{\varphi}^*(df) \,=\, d(f\circ\widetilde{\varphi}) \,=\, X(f\circ\widetilde{\varphi})\,|y|^{-1}dx \,+\, Y(f\circ\widetilde{\varphi})\,y^{-1}dy\,.$$

From (1.24) it follows that

$$\begin{aligned}\widetilde{\varphi}^*(df) \;&=\; \left(\mathrm{sign}\,\varphi'(x)\,X(f)\circ\widetilde{\varphi} \,+\, |y|\frac{\varphi''(x)}{\varphi'(x)}\,Y(f)\circ\widetilde{\varphi}\right)\,|y|^{-1}dx \\ &+\;\; (Y(f)\circ\widetilde{\varphi})\;y^{-1}dy\,.\end{aligned}$$

On substituting in (1.35) the above expression applied to $\widetilde{\varphi}_0^*(df^1)$ as well as to $\widetilde{\varphi}_{01}^*(df^2)$, where $\varphi_{01} := \varphi_1\varphi_0$, one obtains

$$f^0\widetilde{\varphi}_0^*(df^1)\widetilde{\varphi}_0^*\widetilde{\varphi}_1^*(df^2) \,= \tag{1.37}$$

$$\begin{aligned}&=\; f^0\Big(\big(\,\mathrm{sign}\,\varphi_0'(x)\,X(f^1)\circ\widetilde{\varphi}_0 \,+\, |y|\frac{\varphi_0''(x)}{\varphi_0'(x)}\,Y(f^1)\circ\widetilde{\varphi}_0\big)\,Y(f^2)\circ\widetilde{\varphi}_{01} \\ &-\; Y(f^1)\circ\widetilde{\varphi}_0\,\big(\,\mathrm{sign}\,\varphi_{01}'(x)\,X(f^2)\circ\widetilde{\varphi}_{01} \,+\, |y|\frac{\varphi_{01}''(x)}{\varphi_{01}'(x)}\,Y(f^2)\circ\widetilde{\varphi}_{01}\big)\Big)\,\frac{dx\wedge dy}{|y|y}\end{aligned}$$

$$=\; f^0\cdot\mathrm{sign}\,\varphi_0'(x)\cdot X(f^1)\circ\widetilde{\varphi}_0\cdot Y(f^2)\circ\widetilde{\varphi}_{01}\cdot\frac{dx\wedge dy}{|y|y} \tag{1.38}$$

$$-\; f^0\cdot Y(f^1)\circ\widetilde{\varphi}_0\cdot\mathrm{sign}\,\varphi_{01}'(x)\cdot X(f^2)\circ\widetilde{\varphi}_{01}\cdot\frac{dx\wedge dy}{|y|y} \tag{1.39}$$

$$-\; f^0\cdot\left(|y|\frac{\varphi_{01}''(x)}{\varphi_{01}'(x)} \,-\, |y|\frac{\varphi_0''(x)}{\varphi_0'(x)}\right)\cdot Y(f^1)\circ\widetilde{\varphi}_0\cdot Y(f^2)\circ\widetilde{\varphi}_1\widetilde{\varphi}_0\cdot\frac{dx\wedge dy}{|y|y}\,. \tag{1.40}$$

Assuming $\varphi_2\varphi_1\varphi_0 = \mathrm{Id}$ and taking the integral, the left hand side of (1.37) is just

$$TF_\Gamma(f^0U_{\varphi_0^{-1}}, f^1U_{\varphi_1^{-1}}, f^2U_{\varphi_2^{-1}}) \,=\, \int_{FS^1/\mathbb{Z}_2} f^0\widetilde{\varphi}_0^*(df^1)\widetilde{\varphi}_0^*\widetilde{\varphi}_1^*(df^2)\,.$$

On the other hand, after integration each of the three terms in the right hand side of the above can be recognized as being in the image of the characteristic map. Indeed, the term (1.38) is identical to

$$\begin{aligned} &\tau\big(\sigma(f^0 U_{\varphi_0^{-1}})\, X(f^1 U_{\varphi_1^{-1}})\, Y(f^2 U_{\varphi_2^{-1}})\big) = \\ &\quad = \tau\big(f^0 U_{\varphi_0^{-1}}\, \sigma^{-1} X(f^1 U_{\varphi_1^{-1}})\, \sigma^{-1} Y(f^2 U_{\varphi_2^{-1}})\big) \\ &\quad = \chi_\tau(\sigma^{-1} X \otimes \sigma^{-1} Y)(f^0 U_{\varphi_0^{-1}}, f^1 U_{\varphi_1^{-1}}, f^2 U_{\varphi_2^{-1}})\,. \end{aligned}$$

For the second term, using

$$\operatorname{sign} \varphi'_{01}(x) \;=\; \operatorname{sign} \varphi'_1(\varphi_0(x)) \cdot \operatorname{sign} \varphi'_0(x)\,,$$

one identifies (1.39) as

$$\begin{aligned} &\tau\big(\sigma(f^0 U_{\varphi_0^{-1}})\, \sigma Y(f^1 U_{\varphi_1^{-1}})\, X(f^2 U_{\varphi_2^{-1}})\big) = \\ &\quad = \tau\big(f^0 U_{\varphi_0^{-1}}\, Y(f^1 U_{\varphi_1^{-1}})\, \sigma^{-1} X(f^2 U_{\varphi_2^{-1}})\big) \\ &\quad = \chi_\tau(Y \otimes \sigma^{-1} X)(f^0 U_{\varphi_0^{-1}}, f^1 U_{\varphi_1^{-1}}, f^2 U_{\varphi_2^{-1}})\,. \end{aligned}$$

To treat the last term, we note that

$$\begin{aligned} |y|\frac{\varphi''_{01}(x)}{\varphi'_{01}(x)} \;-\; |y|\frac{\varphi''_0(x)}{\varphi'_0(x)} &= |y|\frac{d}{dx}\Big(\log \varphi'_{01}(x)\Big) - |y|\frac{d}{dx}\Big(\log \varphi'_0(x)\Big) \\ = \;\; |y|\frac{d}{dx}\Big(\log \varphi'_1(\varphi_0(x))\Big) &= \operatorname{sign} \varphi'_0(x) |y\varphi'_0(x)| \frac{\varphi''_1(\varphi_0(x))}{\varphi'_1(\varphi_0(x))} \end{aligned}$$

hence (1.40) can be expressed as

$$\begin{aligned} &-\tau\big(\sigma(f^0 U_{\varphi_0^{-1}})\, \delta_1 Y(f^1 U_{\varphi_1^{-1}})\, Y(f^2 U_{\varphi_2^{-1}})\big) = \\ &\quad = -\tau\big(f^0 U_{\varphi_0^{-1}}\, \sigma^{-1}\delta_1 Y(f^1 U_{\varphi_1^{-1}})\, \sigma^{-1} Y(f^2 U_{\varphi_2^{-1}})\big)\,. \end{aligned}$$

To show that $TF^\dagger$ is a Hochschild cocycle we compute $b(TF^\dagger)$, term by term:

$$\begin{aligned} b(-\sigma^{-1}X \otimes \sigma^{-1}Y) \;&=\; -1 \otimes \sigma^{-1}X \otimes \sigma^{-1}Y + \sigma^{-1}X \otimes \sigma^{-1} \otimes \sigma^{-1}Y \\ &+\; 1 \otimes \sigma^{-1}X \otimes \sigma^{-1}Y + \sigma^{-1}\delta_1 \otimes \sigma^{-1}Y \otimes \sigma^{-1}Y \\ &-\; \sigma^{-1}X \otimes \sigma^{-1}Y \otimes \sigma^{-1} - \sigma^{-1}X \otimes \sigma^{-1} \otimes \sigma^{-1}Y \\ &+\; \sigma^{-1}X \otimes \sigma^{-1}Y \otimes \sigma^{-1} = \quad \sigma^{-1}\delta_1 \otimes \sigma^{-1}Y \otimes \sigma^{-1}Y\,, \end{aligned}$$

$$
\begin{aligned}
b(Y \otimes \sigma^{-1}X) &= 1 \otimes Y \otimes \sigma^{-1}X - Y \otimes 1 \otimes \sigma^{-1}X - 1 \otimes Y \otimes \sigma^{-1}X \\
&+ Y \otimes \sigma^{-1}X \otimes \sigma^{-1} + Y \otimes 1 \otimes \sigma^{-1}X + Y \otimes \sigma^{-1}\delta_1 \otimes \sigma^{-1}Y \\
&- Y \otimes \sigma^{-1}X \otimes \sigma^{-1} = \qquad Y \otimes \sigma^{-1}\delta_1 \otimes \sigma^{-1}Y \,,
\end{aligned}
$$

$$
\begin{aligned}
b(\sigma^{-1}\delta_1 Y \otimes \sigma^{-1}Y) &= 1 \otimes \sigma^{-1}\delta_1 Y \otimes \sigma^{-1}Y - \sigma^{-1}\delta_1 Y \otimes \sigma^{-1} \otimes \sigma^{-1}Y \\
&- \sigma^{-1}\delta_1 \otimes \sigma^{-1}Y \otimes \sigma^{-1}Y - Y \otimes \sigma^{-1}\delta_1 \otimes \sigma^{-1}Y \\
&- 1 \otimes \sigma^{-1}\delta_1 Y \otimes \sigma^{-1}Y + \sigma^{-1}\delta_1 Y \otimes \sigma^{-1}Y \otimes \sigma^{-1} \\
&+ \sigma^{-1}\delta_1 Y \otimes \sigma^{-1} \otimes \sigma^{-1}Y - \sigma^{-1}\delta_1 Y \otimes \sigma^{-1}Y \otimes \sigma^{-1} \\
&= -\sigma^{-1}\delta_1 \otimes \sigma^{-1}Y \otimes \sigma^{-1}Y - Y \otimes \sigma^{-1}\delta_1 \otimes \sigma^{-1}Y;
\end{aligned}
$$

summing up, one obtains $\qquad b(TF^\dagger) \,=\, 0.$

Finally, we check the cyclicity of $TF^\dagger$. One has

$$
\begin{aligned}
\tau_2(S(X) \otimes \sigma^{-1}Y) &= S_\delta(S(X))_{(1)}\sigma^{-1}Y \otimes S_\delta(S(X))_{(2)})\sigma^{-1} \\
&= X\sigma^{-1}Y \otimes \sigma^{-1} + Y \otimes X\sigma^{-1} + \delta_1\sigma^{-1}Y \otimes Y\sigma^{-1} \,,
\end{aligned}
$$

$$
\begin{aligned}
\tau_2(Y \otimes \sigma^{-1}X) &= S(Y_{(2)})\sigma^{-1}X \otimes S_\delta(Y_{(1)})\sigma^{-1} \\
&= \sigma^{-1}X \otimes (-Y + 1)\sigma^{-1} - Y\sigma^{-1}X \otimes \sigma^{-1} \\
&= -\sigma^{-1}X \otimes Y\sigma^{-1} + \sigma^{-1}X \otimes \sigma^{-1} - Y\sigma^{-1}X \otimes \sigma^{-1};
\end{aligned}
$$

adding up and taking into account that

$$
\begin{gathered}
X\sigma^{-1}Y \otimes \sigma^{-1} - Y\sigma^{-1}X \otimes \sigma^{-1} + \sigma^{-1}X \otimes \sigma^{-1} \,= \\
-[Y, \sigma^{-1}X] \otimes \sigma^{-1} + \sigma^{-1}X \otimes \sigma^{-1} \,=\, 0 \,,
\end{gathered}
$$

one obtains

$$
\tau_2(TF^\dagger) \,=\, (\sigma^{-1}X - \sigma^{-1}\delta_1 Y) \otimes \sigma^{-1}Y - Y \otimes \sigma^{-1}X \,=\, TF^\dagger \,.
$$

$\square$

We conclude with two comments. The first is that the two periodic Hopf cyclic classes $TF^\dagger$ and $GV^\dagger$ are surely nontrivial, because they lift nontrivial classes. In §4.2 it will be shown that they generate $HP^{\mathrm{ev}}_{(\delta,\sigma^{-1})}(\mathcal{H}_1^\dagger)$ and, respectively, $HP^{\mathrm{odd}}_{(\delta,\sigma^{-1})}(\mathcal{H}_1^\dagger)$.

The second is that not just (δ, σ^{-1}) but actually every pair (δ, σ^k), with $k \in \mathbb{Z}$, is a modular pair in involution for $\mathcal{H}_1^\dagger$. Indeed, the twisted antipode $S_\delta = \delta * S$ is an anti-homomorphism that assumes on the generators the following values:

$$\begin{aligned} S_\delta(\sigma) &= \sigma^{-1}, \qquad & S_\delta(Y) &= -Y + 1, \\ S_\delta(X) &= -\sigma^{-1}(X - \delta_1 Y), \quad & S_\delta(\delta_1) &= -\sigma^{-1}\delta_1 . \end{aligned}$$

One can then easily check that $S_\delta^2 = \mathrm{Id}$; on the other hand $\mathrm{Ad}\,\sigma^k = \mathrm{Id}$, since σ is central. Again in §4.2, it will be proved that $HP^*_{(\delta,\sigma^k)}(\mathcal{H}_1^\dagger) = 0$ for all $k \neq -1$, $k \in \mathbb{Z}$.

1.3 Actions of $\mathcal{H}_1^\dagger$ on modular Hecke algebras

The actions of $\mathcal{H}_1^\dagger$ described in the preceding subsection automatically factor through the double cover $\mathcal{H}_1^{\dagger|2}$. We shall now produce examples of effective actions of the infinite cyclic cover $\mathcal{H}_1^\dagger$ on algebras that arise naturally in the theory of modular forms.

Let us recall from [10] the definition of modular Hecke algebras. These are algebras associated to congruence subgroups Γ of $\mathrm{SL}(2, \mathbb{Z})$, that arise from the fusion of the two quintessential structures coexisting on modular forms, namely the algebra structure given by the pointwise product on the one hand, and the action of the Hecke operators on the other. From the perspective of noncommutative geometry, these algebras describe the holomorphic 'coordinates' of certain noncommutative 'arithmetic spaces', and the action of $\mathcal{H}_1$ is 'elliptic', in the sense that it spans the 'holomorphic tangent space' of the underlying noncommutative space.

Referring to [10] for the occasionally unexplained term in what follows, we denote by $\mathcal{M}$ the algebra of holomorphic modular forms of all levels, on which $G^+(\mathbb{Q}) := \mathrm{GL}^+(2, \mathbb{Q})$ acts via the 'slash' operator

$$f|_k\, g\,(z) \,=\, \det(g)^{\frac{k}{2}}\,(cz+d)^{-k}\, f(g\cdot z)\,, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+(\mathbb{Q})\,.$$

The following variant of the 'slash' operator is more natural when modular forms are viewed as lattice functions:

$$f \dagger_k\; g\,(z) \,=\, (cz+d)^{-k}\, f(g\cdot z) \,=\, (\det g)^{-\frac{k}{2}}\, f|_k\, g\,.$$

Given a congruence subgroup $\Gamma \subset \Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$, the *modular Hecke algebra of level* Γ, to be denoted $\mathcal{A}(\Gamma)$, is the space of all finitely supported maps

$$F : \Gamma\backslash G^+(\mathbb{Q}) \to \mathcal{M}\,, \qquad \Gamma\alpha \mapsto F_\alpha$$

such that

$$F_{\alpha\gamma} \,=\, F_\alpha|\gamma\,, \quad \forall\,\alpha \in G^+(\mathbb{Q})\,, \quad \forall\,\gamma \in \Gamma\,, \tag{1.41}$$

endowed with the product

$$(F^1 * F^2)_\alpha \,:=\, \sum_{\Gamma\beta\in\Gamma\backslash G^+(\mathbb{Q})} F^2_{\alpha\beta^{-1}}|\beta \cdot F^1_\beta\,. \tag{1.42}$$

Any finitely supported map from $\Gamma\backslash G^+(\mathbb{Q})/\Gamma$ to the algebra $\mathcal{M}(\Gamma)$ of modular forms of level Γ trivially fulfills the equation (1.41), but such maps do not exhaust all its solutions. In fact, given $f \in \mathcal{M}$ there exists an $F \in \mathcal{A}(\Gamma)$ such that $F_\alpha \,=\, f$ if and only if $f|\gamma = f\,, \quad \forall\,\gamma \in \Gamma \cap \alpha^{-1}\Gamma\,\alpha$.

To describe the way $\mathcal{H}_1$ acts on modular Hecke algebras (cf. [7, §2]), we recall that while the space of modular forms $\mathcal{M}$ is not invariant under differentiation, there is a classical operator that corrects the derivative, namely

$$X \,=\, \frac{1}{2\pi i}\frac{d}{dz} - \frac{1}{12\pi i}\,\frac{d}{dz}(\log\Delta)\,Y = \frac{1}{2\pi i}\,\frac{d}{dz} - \frac{1}{2\pi i}\,\frac{d}{dz}(\log\eta^4)\,Y\,,$$

where Y is the grading (by the half-weight) operator

$$Y(f) \,=\, \frac{k}{2}\cdot f\,, \quad \forall\,f \in \mathcal{M}_k\,,$$

Δ is the discriminant modular form and η is the Dedekind eta function.

With this choice of a ‘holomorphic connection’, there is a unique action of $\mathcal{H}_1$ on $\mathcal{A}(\Gamma)$ sending the generators of $\mathcal{H}_1$ to the following linear operators on $\mathcal{A}(\Gamma)$:

$$Y(F)_\alpha \,=\, Y(F_\alpha)\,, \quad X(F)_\alpha \,=\, X(F_\alpha)\,, \quad \delta_n(F)_\alpha \,=\, \mu_{n,\alpha}\,F_\alpha\,,$$

where

$$\mu_\alpha\,(z) = \frac{1}{2\pi i}\,\frac{d}{dz}\log\frac{\eta^4|\alpha}{\eta^4}\,, \qquad \mu_{n,\,\alpha} = X^{n-1}(\mu_\alpha), \quad \forall\,n \geq 1\,.$$

Endowed with this action, $\mathcal{A}(\Gamma)$ is an $\mathcal{H}_1$-module algebra, that is

$$h(ab) \,=\, \sum h_{(1)}(a)\, h_{(2)}(b)\,, \quad \forall\, a, b \in \mathcal{A}(\Gamma)\,, \qquad \forall\, h \in \mathcal{H}_1, \tag{1.43}$$

essentially follows from the identity (see [7, Lemma 5]):

$$X(F|\alpha) \,=\, X(F)|\alpha \,+\, \mu_\alpha \cdot Y(F)|\alpha\,, \qquad \alpha \in G^+(\mathbb{Q})\,, \quad f \in \mathcal{M}_k. \tag{1.44}$$

Indeed, using (1.44) in the third line below, one obtains

$$X(F^1 * F^2)_\alpha \,=\, \sum_{\beta\in\Gamma\backslash G^+(\mathbb{Q})} X(F^2_{\alpha\beta^{-1}}|\beta \cdot F^1_\beta) \tag{1.45}$$

$$= \sum_\beta \big(X(F^2_{\alpha\beta^{-1}}|\beta) \cdot F^1_\beta + F^2_{\alpha\beta^{-1}}|\beta \cdot X(F^1_\beta)\big)$$

$$= \sum_\beta \Big(X(F^2_{\alpha\beta^{-1}})|\beta \cdot F^1_\beta + \mu_\beta\, Y(F^2_{\alpha\beta^{-1}})|\beta \cdot F^1_\beta + F^2_{\alpha\beta^{-1}}|\beta \cdot X(F^1_\beta)\Big),$$

that is

$$X(F^1 * F^2) \,=\, X(F^1) * F^2 + F^1 * X(F^2) + \delta_1(F^1) * Y(F^2)\,.$$

In the case of δ_1 one uses the 1-cocycle property of μ_α (that follows from its very definition) to express

$$\begin{aligned} \delta_1(F^1 * F^2)_\alpha &= \mu_\alpha \sum_{\beta\in\Gamma\backslash G^+(\mathbb{Q})} F^2_{\alpha\beta^{-1}}|\beta \cdot F^1_\beta \\ &= (\mu_{\alpha\beta^{-1}}|\beta + \mu_\beta) \sum_\beta F^2_{\alpha\beta^{-1}}|\beta) \cdot F^1_\beta \\ &= \sum_\beta \big((\mu_{\alpha\beta^{-1}} F^2_{\alpha\beta^{-1}})|\beta \cdot F^1_\beta + F^2_{\alpha\beta^{-1}}|\beta \cdot \mu_\beta\, F^1_\beta\big), \end{aligned} \tag{1.46}$$

which means that δ_1 acts as a derivation:

$$\delta_1(F^1 * F^2) \,=\, \delta_1(F^1) * F^2 \,+\, F^1 * \delta_1(F^2)\,. \tag{1.47}$$

We now define a variant of the algebra $\mathcal{A}(\Gamma)$, to be denoted $\mathcal{A}^\dagger(\Gamma)$, obtained by modifying the multiplication rule (1.42) as follows:

$$(F^1 * F^2)_\alpha \,:=\, \sum_{\Gamma\beta\in\Gamma\backslash G^+(\mathbb{Q})} F^2_{\alpha\beta^{-1}} \dagger \beta \cdot F^1_\beta\,, \tag{1.48}$$

The algebra $\mathcal{A}^\dagger(\Gamma)$ carries a natural Hopf module-algebra structure, with respect to the following 'twisted' version of the Hopf algebra $\mathcal{H}_1$.

Proposition 1.2. *There exists a unique action of $\mathcal{H}_1^\dagger$ on $\mathcal{A}^\dagger(\Gamma)$ by which the generators of $\mathcal{H}_1$ act as follows:*

$$\sigma(F)_\alpha \;=\; \det\alpha\cdot F_\alpha\,, \qquad \forall\ \Gamma\alpha\in\Gamma\backslash G^+(\mathbb{Q})\,, \tag{1.49}$$

$$Y(F)_\alpha \;=\; Y(F_\alpha), \quad X(F)_\alpha = X(F_\alpha), \quad \delta_n(F)_\alpha = \mu_{n,\alpha}\,F_\alpha. \tag{1.50}$$

Endowed with this action, $\mathcal{A}^\dagger(\Gamma)$ is a left $\mathcal{H}_1^\dagger$-module algebra.

Proof. The required verifications are similar to those performed in the case of $\mathcal{H}_1$ (cf. [7]). To illustrate the verification of the Hopf action property for X, we remark that when the 'slash' is replaced with the 'dagger' operation, taking into account that $X(f)$ has weight $k+2$, the identity (1.44) becomes

$$X(F\dagger\alpha) \;=\; \det\alpha\cdot X(F)\dagger\alpha \;+\; \mu_\alpha\cdot Y(F)\dagger\alpha\,. \tag{1.51}$$

Using (1.51) instead of (1.44) in the calculation (1.45), one obtains

$$\begin{aligned}
X(F^1*F^2)_\alpha \;&=\; \sum_{\beta\in\Gamma\backslash G^+(\mathbb{Q})} X(F^2_{\alpha\beta^{-1}}\dagger\beta\cdot F^1_\beta)\\
&=\; \sum_\beta \big(X(F^2_{\alpha\beta^{-1}}\dagger\beta)\cdot F^1_\beta + F^2_{\alpha\beta^{-1}}\dagger\beta\cdot X(F^1_\beta)\big)\\
&=\; \sum_\beta \big(\det\beta\, X(F^2_{\alpha\beta^{-1}})\dagger\beta\cdot F^1_\beta\\
&+\; \mu_\beta\, Y(F^2_{\alpha\beta^{-1}})\dagger\beta\cdot F^1_\beta + F^2_{\alpha\beta^{-1}}\dagger\beta\cdot X(F^1_\beta)\big),
\end{aligned}$$

whence

$$X(F^1*F^2) \;=\; X(F^1)*F^2+\sigma(F^1)*X(F^2)+\delta_1(F^1)*Y(F^2)\,.$$

Also, for δ_1, the calculation (1.46) takes the form

$$\begin{aligned}
\delta_1(F^1*F^2)_\alpha \;&=\; \mu_\alpha \sum_{\beta\in\Gamma\backslash G^+(\mathbb{Q})} F^2_{\alpha\beta^{-1}}\dagger\beta\cdot F^1_\beta\\
&=\; \big(\det\beta\,\mu_{\alpha\beta^{-1}}\dagger\beta+\mu_\beta\big)\sum_\beta F^2_{\alpha\beta^{-1}}\dagger\beta\cdot F^1_\beta\\
&=\; \sum_\beta \big((\mu_{\alpha\beta^{-1}}F^2_{\alpha\beta^{-1}})\dagger\beta\cdot\det\beta\,F^1_\beta + F^2_{\alpha\beta^{-1}}\dagger\beta\cdot\mu_\beta\,F^1_\beta\big)\,,
\end{aligned}$$

or equivalently,

$$\delta_1(F^1 * F^2) = \delta_1(F^1) * F^2 + \sigma(F^1) * \delta_1(F^2).$$

$\square$

2 Bicocyclic module associated to a bicrossed product

2.1 MPI coefficients

The Hopf algebra $\mathcal{H}_1^\dagger$ introduced in §1.2 is isomorphic to the 'straight' tensor product $\mathcal{H}_1 \otimes \mathbb{C}[\mathbb{Z}]$ as an algebra but, as we shall see in the next section, it is isomorphic to a cocrossed product $\mathcal{H} \rtimes \mathbb{C}[\mathbb{Z}]$ as a coalgebra. In this section we shall associate a bicocyclic module to such a bicrossed product Hopf algebra, whose diagonal can be identified to the standard cocyclic module defining the Hopf cyclic cohomology. In view of the Getzler-Jones analogue [16] of the Eilenberg-Zilber theorem for bi-paracyclic modules, this makes it possible to compute the desired Hopf cyclic cohomology out of the total complex. This method was used by Getzler and Jones to recover a spectral sequence of Feĭgin and Tsygan [14] for the cyclic homology of crossed products of algebras by discrete groups, and was later generalized by Akbarpour and Khalkhali [1] to the the cyclic homology of crossed products of algebras by Hopf algebras. The key obstacle in implementing this type of construction for Hopf cyclic cohomology resides in the interaction between the antipode and coaction, which is tackled in Lemma 2.3 below. The pay-off for the extra difficulty is that we obtain a bicocyclic module, not just a cylindrical one.

We start with a Hopf algebra $\mathcal{H}$ which admits a left coaction by a commutative Hopf algebra $\mathcal{K}$, $\rho : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$, for which we use the (Sweedler-type) notation

$$\rho(h) = h_{<-1>} \otimes h_{<0>}, \quad \rho^{(k)}(h) = h_{<-k>} \otimes \ldots \otimes h_{<-1>} \otimes h_{<0>},$$

where $\quad \rho^{(k)} = \mathrm{Id}_{\mathcal{K}} \otimes \rho^{(k-1)} \quad$ and $\quad \rho^{(1)} = \rho.$
We call $\mathcal{H}$ a *$\mathcal{K}$-comodule algebra* if for any $h, g \in \mathcal{H}$ the following hold:

$$\rho(hg) = h_{<-1>} g_{<-1>} \otimes h_{<0>} g_{<0>}, \tag{2.1}$$

$$\rho(1) = 1 \otimes 1. \tag{2.2}$$

Similarly, we call $\mathcal{H}$ a $\mathcal{K}$-*comodule coalgebra* if for any $h \in \mathcal{H}$

$$h_{(1)}{}_{<-1>}h_{(2)}{}_{<-1>} \otimes h_{(1)}{}_{<0>} \otimes h_{(2)}{}_{<0>} = h_{<-1>} \otimes h_{<0>}{}_{(1)} \otimes h_{<0>}{}_{(2)}, \tag{2.3}$$

$$\epsilon(h_{<-1>}) \otimes h_{<0>} = 1 \otimes h. \tag{2.4}$$

Definition 2.1. *Let $\mathcal{H}$ be a Hopf algebra which admits a left coaction by a Hopf algebra $\mathcal{K}$. If via this coaction $\mathcal{H}$ is simultaneously a $\mathcal{K}$-comodule algebra and a $\mathcal{K}$-comodule coalgebra, then $\mathcal{H}$ will be called a $\mathcal{K}$-*comodule Hopf algebra*.*

We recall that if $\mathcal{H}$ is a left $\mathcal{K}$-comodule coalgebra then the cocrossed product coalgebra $\mathcal{H} \rtimes \mathcal{K}$ has $\mathcal{H} \otimes \mathcal{K}$ as underlying vector space and the following coalgebra structure:

$$\Delta(h \rtimes k) = h_{(1)} \rtimes h_{(2)}{}_{<-1>}k_{(1)} \otimes h_{(2)}{}_{<0>} \rtimes k_{(2)}, \tag{2.5}$$

$$\epsilon(h \rtimes k) = \epsilon(h)\epsilon(k). \tag{2.6}$$

Lemma 2.2. *The cocrossed product coalgebra $\mathcal{H} \rtimes \mathcal{K}$ admits a Hopf algebra structure via the tensor product algebra structure and with the antipode given by*

$$S(h \rtimes k) = S(h_{<0>}) \rtimes S(h_{<-1>}k).$$

Proof. We need to check that Δ and ϵ are algebra map and that S is convolution inverse of Id. In order to show that Δ is an algebra map we will use the commutativity of $\mathcal{K}$ and also the fact that $\mathcal{H}$ is $\mathcal{K}$-comodule algebra, i.e. (2.1) and (2.2). Thus,

$$\begin{aligned}
&\Delta((h^1 \rtimes k^1)(h^2 \rtimes k^2)) = \Delta(h^1h^2 \rtimes k^1k^2) = \\
&= (h^1{}_{(1)}h^2{}_{(1)}) \rtimes (h^1{}_{(2)}h^2{}_{(2)})_{<-1>}k^1{}_{(1)}k^2{}_{(1)} \otimes (h^1{}_{(2)}h^2{}_{(2)})_{<0>} \rtimes k^1{}_{(2)}k^2{}_{(2)} = \\
&= (h^1{}_{(1)} \rtimes h^2{}_{(2)}{}_{<-1>}k^1{}_{(1)} \otimes h^1{}_{(2)}{}_{<0>} \rtimes k^1{}_{(2)}) \\
&(h^2{}_{(1)} \rtimes h^1{}_{(2)}{}_{<-1>}k^2{}_{(1)} \otimes h^2{}_{(2)}{}_{<0>} \rtimes k^2{}_{(2)}) = \Delta(h^1 \rtimes k^1)\Delta(h^2 \rtimes k^2).
\end{aligned}$$

Similarly, by relying on the fact that $\epsilon_{\mathcal{H}}$ and $\epsilon_{\mathcal{K}}$ are algebra maps, one easily checks that ϵ is algebra maps too.
To verify the antipode axioms, we write

$$\begin{aligned}
&S * \mathrm{Id}(h \rtimes k) = S(h_{(1)} \rtimes h_{(2)}{}_{<-1>}k_{(1)})(h_{(2)}{}_{<0>} \rtimes k_{(2)}) \\
&= S(h_{(1)}{}_{<0>}) \rtimes S(h_{(1)}{}_{<-1>}h_{(2)}{}_{<-1>}k_{(1)})(h_{(2)}{}_{<0>} \rtimes k_{(2)}) \\
&= S(h_{<0>}{}_{(1)}) \rtimes S(h_{<-1>})(h_{<0>}{}_{(2)} \rtimes k_{(2)}) \\
&= S(h_{<0>}{}_{(1)})h_{<0>}{}_{(2)} \rtimes S(h_{<-1>})S(k_{(1)})k_{(2)} = \epsilon(h_{<0>})1_{\mathcal{H}} \rtimes S(h_{<-1>})\epsilon(k) \\
&= \epsilon(h)1_{\mathcal{H}} \rtimes \epsilon(k)1_{\mathcal{K}}.
\end{aligned}$$

A similar computation shows that

$$\mathrm{Id} * S(h \rtimes k) = \epsilon(h \rtimes k) 1_{\mathcal{H}} \rtimes 1_{\mathcal{K}}.$$

$\square$

Lemma 2.3. *If $\mathcal{H}$ is a $\mathcal{K}$-comodule Hopf algebra then the antipode of $\mathcal{H}$ is $\mathcal{K}$-colinear:*

$$h_{<-1>} \otimes S(h_{<0>}) = S(h)_{<-1>} \otimes S(h)_{<0>}, \; h \in \mathcal{H}.$$

Proof. Consider R_1, and R_2 in $\mathrm{Hom}(\mathcal{H}, \mathcal{K} \otimes \mathcal{H})$, defined by:

$$\begin{aligned} & R_1(h) := h_{(1)}{}_{<-1>} h_{(2)}{}_{<-1>} \otimes h_{(1)}{}_{<0>} S(h_{(2)}{}_{<0>}) \,, \\ & R_2(h) = 1 \otimes \epsilon(h). \end{aligned}$$

We first prove that $R_1 = R_2$.
On applying $(1 \otimes m_{\mathcal{H}}) \circ (1 \otimes 1 \otimes S)$, where $m_{\mathcal{H}}$ is the multiplication of $\mathcal{H}$, to both sides of (2.3) one obtains:

$$\begin{aligned} & R_1(h) = h_{(1)}{}_{<-1>} h_{(2)}{}_{<-1>} \otimes h_{(1)}{}_{<0>} S(h_{(2)}{}_{<0>}) = \\ & h_{<-1>} \otimes h_{<0>}{}_{(1)} S(h_{<0>}{}_{(2)}) = \\ & h_{<-1>} \otimes \epsilon(h_{<0>}) = 1 \otimes \epsilon(h) = R_2(h) \end{aligned}$$

To complete the proof it suffices to convolute R_1 and R_2 with the operator $L \in \mathrm{Hom}(\mathcal{H}, \mathcal{K} \otimes \mathcal{H})$,

$$L(h) := S(h)_{<-1>} \otimes S(h)_{<0>}.$$

Indeed, one has

$$h_{<-1>} \otimes S(h_{<0>}) = L * R_1(h) = L * R_2(h) = S(h)_{<-1>} \otimes S(h)_{<0>}.$$

$\square$

Lemma 2.4. *If $\alpha : \mathcal{H} \to \mathbb{C}$ is a $\mathcal{K}$ colinear character, then so is $S_\alpha = \alpha * S$.*

Proof. On the one hand,

$$\begin{aligned} S_\alpha(h)_{<-1>} \otimes S_\alpha(h)_{<-0>} &= \alpha(h_{(1)})S(h_{(2)})_{<-1>} \otimes S(h_{(2)})_{<0>} \\ &= h_{(2)}{}_{<-1>} \otimes \alpha(h_{(1)})S(h_{(2)}{}_{<0>}), \end{aligned}$$

since, by Lemma 2.3, S is colinear.
On the other hand,

$$\begin{aligned} h_{<-1>} \otimes S_\alpha(h_{<0>}) &= h_{<-1>} \otimes \alpha(h_{<0>}{}_{(1)})S(h_{<0>}{}_{(2)}) \\ &= h_{(1)}{}_{<-1>} h_{(2)}{}_{<-1>} \otimes \alpha(h_{(1)}{}_{<0>})S(h_{(2)}{}_{<0>}) \\ &= h_{(2)}{}_{<-1>} \otimes \alpha(h_{(1)})S(h_{(2)}{}_{<0>}), \end{aligned}$$

using the colinearity of α. □

Lemma 2.5. *Let α and β be characters for $\mathcal{H}$ and $\mathcal{K}$ respectively. If α is $\mathcal{K}$ colinear, then*

$$S_{\alpha\otimes\beta}(h \rtimes k) = S_\alpha(h_{<0>}) \rtimes S_\beta(h_{<-1>}k).$$

Proof.

$$\begin{aligned} &S_{\alpha\otimes\beta}(h \rtimes k) = (\alpha\otimes\beta)(h_{(1)} \rtimes h_{(2)}{}_{<-1>}k_{(1)})S(h_{(2)}{}_{<0>} \rtimes k_{(2)}) \\ &= \alpha(h_{(1)})S(h_{(2)}{}_{<0>}) \rtimes \beta(h_{(2)}{}_{<-2>}k_{(1)})S(h_{(2)}{}_{<-1>}k_{(2)}) \\ &= \alpha(h_{(1)})S(h_{(2)}{}_{<0>}) \rtimes S_\beta(h_{(2)}{}_{<-1>})S_\beta(k). \end{aligned}$$

On the other hand,

$$\begin{aligned} &S_\alpha(h_{<0>}) \rtimes S_\beta(h_{<-1>}k) = \alpha(h_{<0>}{}_{(1)})S(h_{<0>}{}_{(2)}) \rtimes S_\beta(h_{<-1>})S_\beta(k) \\ &= \alpha(h_{(1)}{}_{<0>})S(h_{(2)}{}_{<0>}) \rtimes S_\beta(h_{(1)}{}_{<-1>}h_{(2)}{}_{<-1>})S_\beta(k) \\ &= \alpha(h_{(1)})S(h_{(2)}{}_{<0>}) \rtimes S_\beta(h_{(2)}{}_{<-1>})S_\beta(k), \end{aligned}$$

where in the last equality we used once more the colinearity of α. □

Definition 2.6. *Let $\mathcal{H}$ be a $\mathcal{K}$-comodule Hopf algebra. An MPI (α, μ) over $\mathcal{H}$ will be called $\mathcal{K}$-*coinvariant *if α is colinear and $\mu_{<-1>} \otimes \mu_{<0>} = 1 \otimes \mu$. A character β of $\mathcal{K}$ will be called* stable *if $\beta(h_{<-1>})h_{<0>} = h$, for all $h \in \mathcal{H}$.*

Proposition 2.7. *Let (α, μ) and (β, ν) be MPIs for $\mathcal{H}$ and $\mathcal{K}$ respectively. If (α, μ) is coinvariant and β is stable, then the pair $(\alpha \otimes \beta, \mu \rtimes \nu)$ is an MPI for $\mathcal{H} \rtimes \mathcal{K}$.*

Proof. We express

$$\begin{aligned} S^2_{\alpha\otimes\beta}(h \rtimes k) &= S_{\alpha\otimes\beta}(S_\alpha(h_{<0>}) \rtimes S_\beta(h_{<-1>}k)) \\ &= S_\alpha(S_\alpha(h_{<0>})_{<0>}) \rtimes S_\beta(S_\alpha(h_{<0>})_{<-1>} S_\beta(h_{<-1>}k)), \\ &= S^2_\alpha(h_{<0>}) \rtimes S_\beta(h_{<-1>} S_\beta(h_{<-2>})) S^2_\beta(k) \\ &= \mu h_{<0>} \mu^{-1} \rtimes \beta(h_{<-1>}) \nu k \nu^{-1} \\ &= \mu h \mu^{-1} \rtimes \nu k \nu^{-1}, \end{aligned}$$

using first the property that S_α is $\mathcal{K}$-colinear, then the stability of β. ☐

From now on we fix MPIs (α, μ) and (β, ν) satisfying the assumptions of the previous proposition, and thus such that $(\alpha \otimes \beta, \mu \rtimes \nu)$ is an MPI for $\mathcal{H} \rtimes \mathcal{K}$. The rest of this section is devoted to the Hopf cyclic cohomology of $\mathcal{H} \rtimes \mathcal{K}$ with coefficients in the MPI $(\alpha \otimes \beta, \mu \rtimes \nu)$.

For the reader's convenience, we recall that the Hopf cyclic cohomology of a Hopf algebra $\mathcal{H}$ with respect to an MPI (δ, σ) is defined as the cyclic cohomology of the cocyclic module $\mathcal{H}_\natural = \{\mathcal{H}^n_\natural \,;\, n \geq 0\}$, where

$$\mathcal{H}^n_\natural := \mathcal{H}^{\otimes n}, \quad \text{if } n \geq 1 \quad \text{and} \quad \mathcal{H}^0_\natural := \mathbb{C}\,,$$

with the following cocyclic structure:

$$\partial_0(h^1 \otimes \ldots \otimes h^q) = 1 \otimes h^1 \otimes \ldots \otimes h^q \tag{2.7}$$
$$\partial_i(h^1 \otimes \ldots \otimes h^q) = h^1 \otimes \ldots \otimes \Delta(h^i) \otimes \ldots \otimes h^q \tag{2.8}$$
$$\partial_{n+1}(h^1 \otimes \ldots \otimes h^q) = h^1 \otimes \ldots \otimes h^q \otimes \sigma \tag{2.9}$$
$$\sigma_i(h^1 \otimes \ldots \otimes h^q) = h^1 \otimes \ldots \otimes \epsilon(h^{j+1}) \otimes \ldots \otimes h^q \tag{2.10}$$
$$\tau(h^1 \otimes \ldots \otimes h^q) = S_\delta(h^1) \cdot (h^2 \otimes \ldots \otimes h^q \otimes \sigma). \tag{2.11}$$

In what follows, we shall often employ the following shorthand notation:

$$\begin{aligned} &\tilde{h} = h^1 \otimes \ldots \otimes h^q\,, \qquad\qquad \tilde{k} = k^1 \otimes \ldots \otimes k^p\,, \\ &\tilde{h}_{<-1>} \otimes \tilde{h}_{<0>} = (h^1{}_{<-1>} \ldots h^q{}_{<-1>}) \otimes (h^1{}_{<0>} \otimes \ldots \otimes h^q{}_{<0>}). \end{aligned}$$

We now recall that a *bicocyclic module* is a bigraded module whose rows and columns are cocyclic modules and, in addition, vertical operators commute with horizontal operators. It is known, cf. [16], that the diagonal of a bicocyclic module is a cocyclic module.
We define the bicocyclic module $\mathfrak{C} = \{\mathfrak{C}^{(p,q)}\,;\,(p,q) \in \mathbb{Z}^+ \times \mathbb{Z}^+\}$ as follows:

$$\mathfrak{C}^{(p,q)} := \mathcal{K}^p \otimes \mathcal{H}^q\,, \tag{2.12}$$

the horizontal maps

$$\begin{aligned}
&\overrightarrow{\partial}_i : \mathfrak{C}^{(p,q)} \to \mathfrak{C}^{(p+1,q)}, && 0 \le i \le p+1\\
&\overrightarrow{\sigma}_j : \mathfrak{C}^{(p,q)} \to \mathfrak{C}^{(p-1,q)}, && 0 \le j \le p-1\\
&\overrightarrow{\tau} : \mathfrak{C}^{(p,q)} \to \mathfrak{C}^{(p,q)},
\end{aligned}$$

are defined by:

$$\overrightarrow{\partial}_0(\tilde{k} \otimes \tilde{h}) = 1 \otimes k^1 \otimes \ldots \otimes k^p \otimes \tilde{h} \tag{2.13}$$

$$\overrightarrow{\partial}_j(\tilde{k} \otimes \tilde{h}) = k^1 \otimes \ldots \otimes \Delta(k^i) \otimes \ldots \otimes k^p \otimes \tilde{h} \tag{2.14}$$

$$\overrightarrow{\partial}_{p+1}(\tilde{k} \otimes \tilde{h}) = k^1 \otimes \ldots \otimes k^p \otimes \tilde{h}_{<-1>}\nu \otimes \tilde{h}_{<0>} \tag{2.15}$$

$$\overrightarrow{\sigma}_j(\tilde{k} \otimes \tilde{h}) = k^1 \otimes \ldots \otimes \epsilon(k^{j+1}) \otimes \ldots \otimes k^p \otimes \tilde{h} \tag{2.16}$$

$$\overrightarrow{\tau}(\tilde{k} \otimes \tilde{h}) = S_\beta(k^1) \cdot (k^2 \otimes \ldots \otimes k^p \otimes \nu\tilde{h}_{<-1>}) \otimes \tilde{h}_{<0>}; \tag{2.17}$$

the vertical structure is just the cocyclic structure of $\mathcal{H}$, with

$$\uparrow\!\partial_i = \mathrm{Id} \otimes \partial_i : \mathfrak{C}^{(p,q)} \to \mathfrak{C}^{(p,q+1)}, \qquad 0 \le i \le q+1 \tag{2.18}$$

$$\uparrow\!\sigma_j = \mathrm{Id} \otimes \sigma_j : \mathfrak{C}^{(p,q)} \to \mathfrak{C}^{(p,q-1)}, \qquad 0 \le j \le q-1 \tag{2.19}$$

$$\uparrow\!\tau = \mathrm{Id} \otimes \tau : \mathfrak{C}^{(p,q)} \to \mathfrak{C}^{(p,q)}, \tag{2.20}$$

where ∂_i, σ_i, and τ are given by (2.7)-(2.11).

Recall from [17] that *a right-left stable anti Yetter-Drinfeld* (SAYD for short) module over a Hopf algebra $\mathcal{H}$ is a right $\mathcal{H}$-module M equipped with a left coaction of $\mathcal{H}$ such that for any $m \in M$ and any $h \in \mathcal{H}$,

$$\begin{aligned}
m_{<\overline{0}>} m_{<\overline{-1}>} &= m\,,\\
(mh)_{<\overline{-1}>} \otimes (mh)_{<\overline{0}>} &= S(h_{(3)}) m_{<\overline{-1}>} h_{(1)} \otimes m_{<\overline{0}>} h_{<2>};
\end{aligned} \tag{2.21}$$

we have added the bar superscript to the notation as an extra precaution, to avoid any possible confusion in case $\mathcal{H}$ itself is a comodule for some coaction.

The Hopf cyclic cohomology of a Hopf algebra $\mathcal{H}$ with coefficients in a SAYD module M is defined as the cyclic cohomology of the cocyclic module $C^*_H(\mathcal{H}, M) := \{\mathcal{H}^{\otimes n} \otimes M\}_{n\geq 0}$ endowed with the following operators:

$$\begin{aligned}
&\partial_0(h^1 \otimes \ldots \otimes h^n \otimes m) = 1 \otimes h^1 \otimes h^1 \otimes \ldots \otimes h^n \otimes m\\
&\partial_i(h^1 \otimes \ldots \otimes h^n \otimes m) = h^1 \otimes \ldots \otimes \Delta(h^i) \otimes \ldots \otimes h^n \otimes m\\
&\partial_{n+1}(h^1 \otimes \ldots \otimes h^n \otimes m) = h^1 \otimes \ldots \otimes h^n \otimes m_{<\overline{-1}>} \otimes m_{<\bar{0}>}\\
&\sigma_i(h^1 \otimes \ldots \otimes h^n \otimes m) = h^1 \otimes \ldots \otimes \epsilon(h^{i+1}) \otimes \ldots \otimes h^n \otimes m\\
&\tau(h^1 \otimes \ldots \otimes h^n \otimes m) = S(h^1{}_{(2)}) \cdot h^2 \otimes \ldots \otimes h^n \otimes m_{<\overline{-1}>} \otimes m_{<\bar{0}>} h_{(1)}.
\end{aligned}$$

We now equip each $\mathcal{H}^{\otimes q}$ with a right module and left comodule structure over $\mathcal{K}$ via $\tilde{h}k = \beta(k)\tilde{h}$, respectively $\bar{\rho} : \mathcal{H}^{\otimes q} \to \mathcal{K} \otimes \mathcal{H}^{\otimes q}$ defined by

$$\bar{\rho}(\tilde{h}) = \tilde{h}_{<-\bar{1}>} \otimes \tilde{h}_{<\bar{0}>} := \tilde{h}_{<-1>}\nu \otimes \tilde{h}_{<0>}.$$

Lemma 2.8. *For each $q \geq 0$, $\mathcal{H}^{\otimes q}$ is an SAYD module over $\mathcal{K}$ and the qth row of* (2.12) *is isomorphic to the cocyclic module $\{C^p_{\mathcal{K}}(\mathcal{K}, \mathcal{H}^q)\}_{p\geq 0}$.*

Proof. The second part of the lemma is clear, so we only need to prove the first statement. Using the commutativity of $\mathcal{K}$ and also the fact that (δ, ν) is a MPI for $\mathcal{K}$, one gets

$$\bar{\rho}(\tilde{h}k) = \beta(k)\bar{\rho}(\tilde{h}) = \beta(k)\tilde{h}_{<-1>}\nu \otimes \tilde{h}_{<0>} = S(k_{(3)})\tilde{h}_{<-1>}\nu k_{(1)} \otimes \beta(k_{(2)})\tilde{h}_{<0>},$$

which shows that $\mathcal{H}^{\otimes q}$ is a anti Yetter-Drinfeld module over $\mathcal{K}$. To prove stability we use the extra property of β, i.e. $\beta(h_{<-1>})h_{<0>} = h$. $\square$

Proposition 2.9. *The bigraded module $\mathfrak{C}$ defined by* (2.12) *is bicocyclic.*

Proof. First we show that each row and column is a cocyclic module. For columns this is obvious because the operators are identical to those of the cocyclic module $\mathcal{H}_\natural$. On the other hand, Lemma 2.8 shows that the rows are cocyclic module too.
It remains to show that the vertical operators and the horizontal operators commute. To this end, it suffices to prove that $\overrightarrow{\tau}$ commutes with all vertical operators. We only show that $\overrightarrow{\tau}$ and $\uparrow\tau$ commute with one another and leave to the reader to check the rest. Using Lemma 2.3, Lemma 2.4, the fact that μ is $\mathcal{K}$-coinvariant, (2.1) and (2.3), we can write:

$$\begin{aligned}
&\overrightarrow{\tau}\circ \uparrow\tau(\tilde{k}\otimes\tilde{h}) = \overrightarrow{\tau}(\tilde{k}\otimes S_\alpha(h^1)\cdot h^2\otimes\ldots\otimes h^q\otimes\mu)\\
&= S_\beta(k^1)\cdot(k^2\otimes\ldots\otimes k^p\otimes\nu S(h^1{}_{(q+1)})_{<-1>}h^2{}_{<-1>}\cdots\\
&\ldots S(h^1{}_{(2)})_{<-1>}h^q{}_{<-1>}\otimes S_\alpha(h^1{}_{(1)})_{<-1>}\mu_{<-1>})\otimes S(h^1{}_{(q+1)})_{<0>}h^2{}_{<0>}\otimes\ldots\\
&\ldots\otimes S(h^1{}_{(2)})_{<0>}h^q{}_{<0>}S_\alpha(h^1{}_{(1)})_{<0>}\mu_{<0>} =\\
&= S_\beta(k^1)\cdot(k^2\otimes\ldots\otimes k^p\otimes\nu\tilde{h}_{<-1>})\otimes S_\alpha(h^1{}_{<0>})\cdot(h^2{}_{<0>}\otimes\ldots\otimes h^q{}_{<0>}\otimes\mu)\\
&=\uparrow\tau\circ\overrightarrow{\tau}(\tilde{k}\otimes\tilde{h}).
\end{aligned}$$

□

Let $\mathfrak{D}$ be the diagonal of $\mathfrak{C}$, that is $\mathfrak{D}=\{\mathfrak{D}^n\,;\,n\geq 0\}$ with

$$\mathfrak{D}^n := \mathcal{K}^{\otimes n}\otimes\mathcal{H}^{\otimes n}\equiv\mathcal{K}^{\otimes n}\otimes\mathcal{H}^{\otimes n},\quad \text{if } n\geq 1\quad\text{and}\quad \mathfrak{D}^0:=\mathbb{C}\,,$$

equipped with the following cocyclic structure:

$$\begin{aligned}
&d_0(\tilde{k}\otimes\tilde{h}) = 1\otimes k^1\otimes\ldots\otimes k^n\otimes 1\otimes h^1\otimes\ldots\otimes h^n,\\
&d_i(\tilde{k}\otimes\tilde{h}) = k^1\otimes\ldots\otimes\Delta(k^i)\otimes\ldots\otimes k^n\otimes h^1\ldots\otimes\Delta(h^i)\otimes\ldots\otimes h^n,\\
&d_{n+1}(\tilde{k}\otimes\tilde{h}) = k^1\otimes\ldots\otimes k^n\otimes\nu\tilde{h}_{<-1>}\otimes h^1{}_{<0>}\otimes\ldots\otimes h^n{}_{<0>}\otimes\mu,\\
&s_j(\tilde{k}\otimes\tilde{h}) = k^1\otimes\ldots\otimes\epsilon(k^{j+1})\otimes\ldots\otimes k^n\otimes h^1\ldots\otimes\epsilon(h^{j+1})\otimes\ldots\otimes h^n,\\
&t(\tilde{k}\otimes\tilde{h}) = S_\beta(k^1)\cdot(k^2\otimes\ldots\\
&\ \ldots\otimes k^n\otimes\nu\tilde{h}_{<-1>})\otimes S_\alpha(h^1{}_{<0>})\cdot(h^2{}_{<0>}\otimes\ldots\otimes h^n{}_{<0>}\otimes\mu).
\end{aligned}$$

We now proceed to identify the cocyclic module $(\mathcal{H}\rtimes\mathcal{K})_\natural$ with $\mathfrak{D}$. To this end we introduce the maps Ψ and its inverse Ψ^{-1}. First, we define $\Psi:(\mathcal{H}\rtimes\mathcal{K})_\natural\longrightarrow\mathfrak{D}$, by setting

$$\Psi=\prod_{i=1}^{n}\mathrm{Id}_{\mathcal{K}^{\otimes n-i}}\otimes\top^{\otimes i}\otimes\mathrm{Id}_{\mathcal{H}^{\otimes n-i}}, \tag{2.22}$$

where

$$\top : \mathcal{K} \otimes \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}, \quad \top(h \otimes k) = h_{<-1>} k \otimes h_{<0>}. \tag{2.23}$$

The effect of Ψ is explicitly seen as follows:

$$\Psi(h^1 \rtimes k^1 \otimes \ldots \otimes h^n \rtimes k^n) = h^1{}_{<-n>} k^1 \otimes h^1{}_{<-n+1>} h^2{}_{<-n+1>} k^2 \otimes \ldots$$
$$\ldots \otimes h^1{}_{<-2>} \ldots h^{n-1}{}_{<-2>} k^{n-1} \otimes h^1{}_{<-1>} \ldots h^n{}_{<-1>} k^n \otimes h^1{}_{<0>} \otimes \ldots \otimes h^n{}_{<0>}.$$

The inverse $\Psi^{-1} : \mathfrak{D} \longrightarrow (\mathcal{H} \rtimes \mathcal{K})_\natural$ is given by a similar expression,

$$\Psi^{-1} = \prod_{i=0}^{n-1} id_{\mathcal{K}^{\otimes i}} \otimes \perp^{\otimes n-i} \otimes id_{\mathcal{H}^{\otimes i}}, \tag{2.24}$$

where

$$\perp : \mathcal{K} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}, \quad \perp(k \otimes h) = h_{<0>} \otimes S(h_{<-1>}) k. \tag{2.25}$$

Its explicit expression is:

$$\Psi^{-1}(\tilde{k} \otimes \tilde{h}) =$$
$$h^1{}_{<0>} \rtimes S(h^1{}_{<-1>}) k^1 \otimes \ldots \otimes h^n{}_{<0>} \rtimes S(h^1{}_{<-n>} h^2{}_{<-n+1>} \ldots h^n{}_{<-1>}) k^n.$$

In order to check that Ψ and Ψ^{-1} are in fact inverse of one another, we only need to show that $\top$ and $\perp$ are so. Indeed,

$$\top\perp(k \otimes h) = \top(h_{<0>} \otimes S(h_{<-1>}) k) = h_{<-1>} S(h_{<-2>}) k \otimes h_{<0>} = k \otimes h.$$
$$\perp\top(h \otimes k) = \perp(h_{<-1>} k \otimes h_{<0>}) = h_{<0>} \otimes S(h_{<-1>}) h_{<-2>} k = h \otimes k.$$

Proposition 2.10. *The maps Ψ and Ψ^{-1} are cyclic.*

Proof. To show that Ψ is a cyclic map, we first use Lemma 2.5 to observe that:

$$\Delta^n(S_{(\alpha\otimes\beta)}(h_{<0>} \rtimes S(h_{<-1>} k))) = S(h_{(n+1)}) \rtimes h_{(n)}{}_{<-1>} \ldots h_{(1)}{}_{<-n>} S(k_{(n+1)}) \otimes$$
$$\otimes S(h_{(n)}{}_{<0>}) \rtimes h_{(n-1)}{}_{<-1>} \ldots h_{(1)}{}_{<-n+1>} S(k_{(n)}) \otimes \ldots \otimes S(h_{(2)}{}_{<0>}) \rtimes h_{(1)}{}_{<-1>} S(k_{(2)})$$
$$\otimes S_\alpha(h_{(1)}{}_{<0>}) \rtimes S_\beta(k_{(1)}).$$

In view of the above identity we can write

$$
\begin{aligned}
&\tau\Psi^{-1}(\tilde{k}\otimes\tilde{h}) = \\
&= \tau(h^1{}_{<0>} \rtimes S(h^1{}_{<-1>})k^1 \otimes \ldots \otimes h^n{}_{<0>} \rtimes S(h^1{}_{<-n>}h^2{}_{<-n+1>} \ldots h^n{}_{<-1>})k^n) \\
&= S_{\alpha\otimes\beta}(h^1{}_{<0>} \rtimes S(h^1{}_{<-1>}k^1) \cdot (h^2{}_{<0>} \rtimes S(h^1{}_{<-2>}h^2{}_{<-1>})k^2 \otimes \ldots \\
&\ldots \otimes h^n{}_{<0>} \rtimes S(h^1{}_{<-n>}h^2{}_{<-n+1>} \ldots h^n{}_{<-1>})k^n \otimes \mu \rtimes \nu) = \\
&= \Delta(S_{\alpha\otimes\beta}(h^1{}_{<0>} \rtimes S(h^1{}_{<-1>}k^1))(h^2{}_{<0>} \rtimes S(h^1{}_{<-2>}h^2{}_{<-1>})k^2 \otimes \ldots \\
&\ldots \otimes h^n{}_{<0>} \rtimes S(h^1{}_{<-n>}h^2{}_{<-n+1>} \ldots h^n{}_{<-1>})k^n \otimes \mu \rtimes \nu) = \\
&= S(h^1{}_{<0>(n+1)})h^2{}_{<0>} \rtimes h^1{}_{<0>(n)<-1>} \ldots h^1{}_{<0>(1)<-n>} S(k^1{}_{(n+1)})S(h^1{}_{<-2>}h^2{}_{<-1>}) \otimes \ldots \\
&\ldots \otimes S(h^1{}_{<0>(2)<0>})h^n{}_{<0>} \rtimes h^1{}_{<0>(1)<-1>} S(k^1{}_{(2)})S(h^1{}_{<-n>}h^2{}_{<-n+1>} \ldots h^n{}_{<-1>})k^n \otimes \\
&\otimes S_\alpha(h^1{}_{<0>(1)<0>})\mu \rtimes S_\beta(k^1{}_{(1)})\nu.
\end{aligned}
$$

Using the fact that $\Delta^{(n)} : \mathcal{H} \to \mathcal{H}^{\otimes n+1}$, S, and S_α are $\mathcal{K}$-colinear, and $\mathcal{K}$ is commutative, one has:

$$
\begin{aligned}
&\Psi\tau\Psi^{-1}(\tilde{k}\otimes\tilde{h}) = h^1{}_{<0>(n+1)<-n>} \ldots h^1{}_{<0>(1)<-n>} h^2{}_{<0><-n>} \\
&S(h^1{}_{<-1>}h^2{}_{<-1>})S(k^1{}_{(n+1)})k^2 \otimes h^1{}_{<0>(n+1)<-n+1>} \ldots \\
&\ldots h^1{}_{<0>(1)<-n+1>} h^2{}_{<0><-n+1>} h^3{}_{<0><-n+1>} \\
&S(h^1{}_{<-2>}h^2{}_{<-2>}h^3{}_{<-1>})S(k^1{}_{(n)})k^3 \otimes \\
&\otimes \ldots \otimes h^1{}_{<0>(n+1)<-2>} \ldots h^1{}_{<0>(1)<-2>} h^2{}_{<0><-2>} \ldots h^n{}_{<0><-2>} \\
&S(h^1{}_{<-n+1>}h^2{}_{<-n+1>} \ldots h^n{}_{<-1>})S(k^1{}_{(2)})k^n \\
&\otimes h^1{}_{<0>(n+1)<-1>} \ldots h^1{}_{<0>(1)<-1>} h^2{}_{<0><-1>} \ldots h^n{}_{<0><-1>} S_\beta(k^1{}_{(1)})\nu \otimes \\
&S(h^1{}_{<0>(n+1)<0>})h^2{}_{<0><0>} \otimes \ldots \otimes S(h^1{}_{<0>(1)<0>})\mu \\
&= h^1{}_{<-n>}h^2{}_{<-n>} S(h^1{}_{<-n-1>}h^2{}_{<-n-1>})S(k^1{}_{(n+1)})k^2 \otimes \ldots \\
&\ldots \otimes h^1{}_{<-2>} \ldots h^n{}_{<-2>} S(h^1{}_{<1-2n>}h^2{}_{<1-2n>}h^3{}_{<3-2n>} \ldots h^n{}_{<-3>})S(k^1{}_{(2)})k^n \\
&\otimes h^1{}_{<-1>} \ldots h^n{}_{<-1>} S_\beta(k^1{}_{(1)})\nu \otimes S(h^1{}_{<0>(n+1)})h^2{}_{<0>} \otimes \ldots \\
&\ldots \otimes S(h^1{}_{<0>(2)})h^n{}_{<0>} \otimes S_\alpha(h^1{}_{<0>(1)})\mu = \\
&= S(k^1{}_{(n+1)}) \otimes \ldots \otimes S(k^1{}_{(2)}) \otimes S_\beta(k^1{}_{(1)})\nu\tilde{h}_{<-1>} \otimes S(h^1{}_{<0>(n+1)})h^2{}_{<0>} \otimes \\
&\ldots \otimes S(h^1{}_{<0>(2)})h^n{}_{<0>} \otimes S_\alpha(h^1{}_{<0>(1)})\mu = t(\tilde{k}\otimes\tilde{h}).
\end{aligned}
$$

By a similar computation one shows that $\Psi\partial_i\Psi^{-1} = d_i$ and $\Psi\sigma_i\Psi^{-1} = s_i$. $\square$

Let $Tot(\mathfrak{C})$ the corresponding total mixed complex

$$Tot(\mathfrak{C})_n = \bigoplus_{p+q=n} \mathcal{K}^p \otimes \mathcal{H}^q \,.$$

We shall denote by $\mathrm{Tot}(\mathfrak{C})$ the associated *normalized* subcomplex, obtained by retaining only the elements annihilated by all degeneracy operators. The total boundary $b_T + B_T$ defined by

$$\overrightarrow{b}_p = \sum_{i=0}^{p+1} (-1)^i \overrightarrow{\partial}_i, \qquad \uparrow b_q = \sum_{i=0}^{q+1} (-1)^i \uparrow\partial_i, \tag{2.26}$$

$$b_T = \sum_{p+q=n} \overrightarrow{b}_p + \uparrow b_q, \tag{2.27}$$

$$\overrightarrow{B}_p = (\sum_{i=0}^{p-1} (-1)^{(p-1)i} \overrightarrow{\tau}^i) \overrightarrow{\sigma}_{p-1} \overrightarrow{\tau}, \quad \uparrow B_q = (\sum_{i=0}^{q-1} (-1)^{(q-1)i} \uparrow\tau^i) \overrightarrow{\sigma}_{q-1} \uparrow\tau, \tag{2.28}$$

$$B_T = \sum_{p+q=n} \overrightarrow{B}_p + \uparrow B_q. \tag{2.29}$$

The total complex of a bicocyclic module is a mixed complex, and so its cyclic cohomology is well-defined. In view of the analogue of the Eilenberg-Zilber theorem for bi-paracyclic modules [16], we can assert that the diagonal mixed complex $(\mathfrak{D}, b, B)$ and the total mixed complex $(\mathrm{Tot}\,\mathfrak{C}, b_T, B_T)$ are quasi-isomorphic in Hochschild and cyclic cohomology, via Alexander-Whitney map

$$AW := \bigoplus_{p+q=n} AW_{p,q} : \mathrm{Tot}(\mathfrak{C})^n \to \mathfrak{D}^n, \tag{2.30}$$

$$AW_{p,q} : \mathcal{K}^{\otimes p} \otimes \mathcal{H}^{\otimes q} \longrightarrow \mathcal{K}^{\otimes p+q} \otimes \mathcal{H}^{\otimes p+q}$$

$$AW_{p,q} = (-1)^{p+q} \underbrace{\uparrow\partial_0 \uparrow\partial_0 \ldots \uparrow\partial_0}_{p\,\text{times}} \overrightarrow{\partial}_n \overrightarrow{\partial}_{n-1} \ldots \overrightarrow{\partial}_{p+1}\,.$$

Together with Proposition 2.10, this allows to conclude with the main result of this section.

Theorem 2.11. *The mixed complex* $((\mathcal{H} \rtimes \mathcal{K})_\natural, b, B)$ *is quasi-isomorphic to the mixed complex* $(\mathrm{Tot}\,\mathfrak{C}, b_T, B_T)$.

2.2 SAYD coefficients

With a view towards future applications of the methods of the present paper, we now digress to extend the results of the previous subsection to the general

framework of Hopf cyclic cohomology for module coalgebras with coefficients in SAYD modules (cf. [17, 18]). The allowable SAYD coefficients are those that satisfy appropriate compatibility conditions, matching the cocrossed product construction of the Hopf algebra itself.

With $\mathcal{H}$ and $\mathcal{K}$ satisfying the same assumptions as above, we let C and D denote a left $\mathcal{H}$-module coalgebra, respectively a left $\mathcal{K}$-module coalgebra. In addition we assume that C is left $\mathcal{K}$-comodule coalgebra via

$$\nabla : C \to \mathcal{K} \otimes C, \tag{2.31}$$
$$\nabla(c) = c_{<-1>} \otimes c_{<0>}. \tag{2.32}$$

One can then define the coalgebra $C \rtimes D$, with $C \otimes D$ as underlying vector a space and the following coalgebra structure:

$$\Delta(c \rtimes d) = c_{(1)} \otimes c_{(2)}{}_{<-1>} d_{(1)} \otimes c_{(2)}{}_{<0>} \rtimes d_{(2)} \tag{2.33}$$
$$\epsilon(c \rtimes d) = \epsilon(c)\epsilon(d). \tag{2.34}$$

It is straightforward to check that $C \rtimes D$ is a counital coassociative coalgebra. In addition we shall assume that C satisfies the following compatibility condition, relating the coaction of $\mathcal{K}$ and the action of $\mathcal{H}$:

$$\nabla(hc) = h_{<-1>} c_{<-1>} \otimes h_{<0>} c_{<0>}, \quad \forall h \in \mathcal{H},\, c \in C. \tag{2.35}$$

One endows $C \rtimes D$ with a left action of $\mathcal{H} \rtimes \mathcal{K}$ by defining, for any $h \in \mathcal{H}$, $k \in \mathcal{K}$, $c \in C$ and $d \in D$,

$$(h \rtimes k)(c \rtimes d) = hc \rtimes kd. \tag{2.36}$$

Proposition 2.12. *The action defined by* (2.36) *makes* $C \rtimes D$ *a* $\mathcal{H} \rtimes \mathcal{K}$ *module coalgebra.*

Proof. Using (2.33), (2.35), (2.3), and the fact that K is commutative, we have:

$$\begin{aligned}
&\Delta((h \rtimes k)(c \rtimes d)) = \Delta(hc \rtimes kd) = \\
&(hc)_{(1)} \rtimes (hc)_{(2)}{}_{<-1>} (kd)_{(1)} \otimes (hc)_{(2)}{}_{<0>} \otimes (kd)_{(2)} = \\
&h_{(1)} c_{(1)} \rtimes h_{(2)}{}_{<-1>} c_{(2)}{}_{<-1>} k_{(1)} d_{(1)} \otimes h_{(2)}{}_{<0>} c_{(2)}{}_{<0>} \otimes k_{(2)} d_{(2)} = \\
&(h \rtimes k)\Delta(c \rtimes d).
\end{aligned}$$

∎

Now let M and N be right-left SAYD modules over $\mathcal{H}$ and $\mathcal{K}$ respectively.

Definition 2.13. *We say that the SAYD module M is $\mathcal{K}$-coinvariant if for any $h \in \mathcal{H}$, and any $m \in M$:*

$$1) \quad h_{<-1>} \otimes mh_{<0>} = 1 \otimes mh, \tag{2.37}$$

$$2) \quad m_{<\overline{-1}><-1>} \otimes m_{<\overline{-1}><0>} \otimes m_{<\overline{0}>} = 1 \otimes m_{<\overline{-1}>} \otimes m_{<\overline{0}>} \tag{2.38}$$

Definition 2.14. *We say that the SAYD module N is $\mathcal{H}$-stable if for any $h \in \mathcal{H}$, and $n \in N$ one has:*

$$nh_{<-1>} \otimes h_{<0>} = n \otimes h \tag{2.39}$$

We now endow $M \otimes N$ the a right action and left coaction of $\mathcal{H} \rtimes \mathcal{K}$ as follows

$$(M \otimes N) \otimes \mathcal{H} \rtimes \mathcal{K} \to (M \otimes N), \tag{2.40}$$

$$(m \otimes n)(h \rtimes k) = mh \otimes nk \tag{2.41}$$

$$(M \otimes N) \to \mathcal{H} \rtimes \mathcal{K} \otimes (M \otimes N), \tag{2.42}$$

$$(m \otimes n) \mapsto m_{<\overline{-1}>} \rtimes \otimes n_{<\overline{-1}>} \otimes (m_{<\overline{0}>} \otimes n_{<\overline{0}>}) \tag{2.43}$$

Proposition 2.15. *Let M be $\mathcal{K}$-coinvariant and N be $\mathcal{H}$-stable. Then via the action and coaction defined by* (2.40) *and* (2.42) *respectively, $M \otimes N$ is an SAYD module over $\mathcal{H} \rtimes \mathcal{K}$.*

Proof. First we show that $M \otimes N$ is a comodule over $\mathcal{H} \rtimes \mathcal{K}$.
Indeed, using (2.38) and the fact that M and N are comodule over $\mathcal{H}$, and $\mathcal{K}$ respectively, one has

$$\begin{aligned}
&(m \otimes n)_{<\overline{-1}>^{(1)}} \otimes (m \otimes n)_{<\overline{-1}>^{(2)}} \otimes (m \otimes n)_{<\overline{0}>} = \\
&(m_{<\overline{-1}>} \rtimes n_{<\overline{-1}>})_{(1)} \otimes (m_{<\overline{-1}>} \rtimes \otimes n_{<\overline{-1}>})_{(2)} \otimes m_{<\overline{0}>} \otimes n_{<\overline{0}>} = \\
&m_{<\overline{-1}>^{(1)}} \rtimes m_{<\overline{-1}>^{(2)}<-1>} n_{<\overline{-1}>^{(1)}} \otimes m_{<\overline{-1}>^{(2)}<0>} \rtimes n_{<\overline{-1}>^{(2)}} \otimes m_{<\overline{0}>} \otimes n_{<\overline{0}>} = \\
&m_{<\overline{-2}>} \rtimes m_{<\overline{-1}><-1>} n_{<\overline{-1}>^{(1)}} \otimes m_{<\overline{-1}><0>} \rtimes n_{<\overline{-1}>} \otimes m_{<\overline{0}>} \otimes n_{<\overline{0}>} = \\
&m_{<\overline{-2}>} \rtimes n_{<\overline{-2}>} \otimes m_{<\overline{-1}>} \rtimes n_{<\overline{-1}>} \otimes m_{<\overline{0}>} \otimes n_{<\overline{0}>} = \\
&(m \otimes n)_{<\overline{-2}>} \otimes (m \otimes n)_{<\overline{-1}>} \otimes (m \otimes n)_{<\overline{0}>}.
\end{aligned}$$

Let us prove that $M \otimes N$ satisfies (2.21). First of all one sees that

$$(\Delta \otimes id)(\Delta(h \rtimes k)) =$$
$$h_{(1)} \rtimes h_{(2)}{}_{<-1>} h_{(3)}{}_{<-2>} k_{(1)} \otimes h_{(2)}{}_{<0>} \rtimes h_{(3)}{}_{<-1>} k_{(2)} \otimes h_{(3)}{}_{<0>} \rtimes k_{(3)}.$$

Using the above identity, the fact that both M and N satisfy (2.21), that $\mathcal{K}$ is commutative, that M satisfies (2.39), and that N satisfies (2.37), one can write successively

$$S((h \rtimes k)_{(3)})(m \otimes n)_{<\overline{-1}>}(h \rtimes k)_{(1)} \otimes (m \otimes n)_{<0>}(h \rtimes k)_{<2>} =$$
$$S(h_{(3)}{}_{<0>} \rtimes k_{(3)})(m_{<\overline{-1}>} \rtimes n_{<\overline{-1}>})(h_{(1)} \rtimes h_{(2)}{}_{<-1>} h_{(3)}{}_{<-2>} k_{(1)}) \otimes$$
$$\otimes (m_{<\overline{0}>} \otimes n_{<\overline{0}>})(h_{(2)}{}_{<0>} \rtimes h_{(3)}{}_{<-1>} k_{(2)}) =$$
$$S(h_{(3)}{}_{<0>}) m_{<\overline{-1}>} h_{(1)} \rtimes S(h_{(3)}{}_{<-1>} k_{(3)}) n_{<\overline{-1}>} h_{(2)}{}_{<-1>} h_{(3)}{}_{<-3>} k_{(1)} \otimes$$
$$\otimes m_{<\overline{0}>} h_{(2)}{}_{<0>} \otimes n_{<\overline{0}>} h_{(3)}{}_{<-2>} k_{(2)} =$$

$$S(h_{(3)}{}_{<0>}) m_{<\overline{-1}>} h_{(1)} \rtimes S(h_{(3)}{}_{<-1>} k_{(3)}) n_{<\overline{-1}>} h_{(3)}{}_{<-2>} k_{(1)} \otimes$$
$$\otimes m_{<\overline{0}>} h_{(2)} \otimes n_{<\overline{0}>} k_{(2)} =$$
$$S(h_{(3)}{}_{<0>}) m_{<\overline{-1}>} h_{(1)} \rtimes S(h_{(3)}{}_{<-1>}) h_{(3)}{}_{<-2>} S(k_{(3)}) n_{<\overline{-1}>} k_{(1)} \otimes$$
$$\otimes m_{<\overline{0}>} h_{(2)} \otimes n_{<\overline{0}>} k_{(2)} =$$
$$S(h_{(3)}) m_{<\overline{-1}>} h_{(1)} \rtimes S(k_{(3)}) n_{<\overline{-1}>} k_{(1)} \otimes m_{<\overline{0}>} h_{(2)} \otimes n_{<\overline{0}>} k_{(2)} =$$
$$(mh)_{<\overline{-1}>} \rtimes (nk)_{<\overline{-1}>} \otimes (mh)_{<\overline{0}>} \otimes (nk)_{<\overline{0}>}$$
$$= [(m \otimes n)(h \rtimes k)]_{<\overline{-1}>} \otimes [(m \otimes n)(h \rtimes k)]_{<\overline{0}>}.$$

It is easily seen that if M and N are stable, then so is $M \otimes N$. $\square$

For convenience, we recall (see [18]) the definition of Hopf cyclic cohomology of a left $\mathcal{H}$-module coalgebra with coefficients in a SAYD module M. It is the cyclic cohomology of the cocyclic module defined by

$$\mathcal{C}^n = C^n_H(C, M) = M \otimes_H C^{\otimes n+1}, \quad n \geq 0,$$

with the following cocyclic structure, where we abbreviate $\tilde{c} = c^0 \otimes \ldots \otimes c^n$,

$$\partial_i(m \otimes_{\mathcal{H}} \tilde{c}) = m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes \Delta(c_i) \otimes \ldots \otimes c^n, \tag{2.44}$$
$$\partial_{n+1}(m \otimes_{\mathcal{H}} \tilde{c}) = m_{<\overline{0}>} \otimes_{\mathcal{H}} c^0{}_{(2)} \otimes c^1 \otimes \ldots \otimes c^n \otimes m_{<\overline{-1}>} c^0{}_{(1)}, \tag{2.45}$$
$$\sigma_i(m \otimes_{\mathcal{H}} \tilde{c}) = m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes \epsilon(c^{i+1}) \otimes \ldots \otimes c^n, \tag{2.46}$$
$$\tau(m \otimes_{\mathcal{H}} \tilde{c}) = m_{<\overline{0}>} \otimes_{\mathcal{H}} c^1 \otimes \ldots \otimes c^n \otimes m_{<\overline{-1}>} c^0. \tag{2.47}$$

We define the bigraded module $\mathfrak{X}^{p,q}$ by

$$\mathfrak{X}^{p,q} = N \otimes_{\mathcal{K}} D^{\otimes p+1} \otimes M \otimes_{\mathcal{H}} \otimes C^{\otimes q+1},$$

and endow $\mathfrak{X}$ wit the operators

$$\overrightarrow{\partial}_i : \mathfrak{X}^{(p,q)} \to \mathfrak{X}^{(p+1,q)}, \qquad 0 \le i \le p+1 \tag{2.48}$$

$$\overrightarrow{\sigma}_j : \mathfrak{X}^{(p,q)} \to \mathfrak{X}^{(p-1,q)}, \qquad 0 \le j \le p-1 \tag{2.49}$$

$$\overrightarrow{\tau} : \mathfrak{X}^{(p,q)} \to \mathfrak{X}^{(p,q)}, \tag{2.50}$$

defined by

$$\overrightarrow{\partial}_j(n \otimes \tilde{d} \otimes m \otimes \tilde{c}) = n \otimes c^0 \otimes \ldots \otimes \Delta(c^i) \otimes \ldots \otimes c^p \otimes m \otimes \tilde{d}, \tag{2.51}$$

$$\overrightarrow{\partial}_{p+1}(n \otimes \tilde{d} \otimes m \otimes \tilde{c}) = \tag{2.52}$$

$$n_{<\overline{0}>} \otimes d^0{}_{(2)} \otimes \ldots \otimes d^p \otimes \tilde{c}_{<-1>} n_{<\overline{-1}>} d^0{}_{(1)} \otimes m \otimes \tilde{c}_{<0>},$$

$$\overrightarrow{\sigma}_j(n \otimes \tilde{d} \otimes m \otimes \tilde{c}) = n \otimes d^0 \otimes \ldots \otimes \epsilon(d^{j+1}) \otimes \ldots \otimes d^p \otimes \tilde{c}, \tag{2.53}$$

$$\overrightarrow{\tau}(n \otimes \tilde{d} \otimes m \otimes \tilde{c}) = \tag{2.54}$$

$$n_{<\overline{0}>} \otimes d^1 \otimes \ldots \otimes d^p \otimes \tilde{c}_{<-1>} n_{<\overline{-1}>} d^0 \otimes m \otimes \tilde{c}_{<0>};$$

the vertical structure is just the cocyclic structure of $\mathcal{C}^*$, with

$$\uparrow\partial_i = \mathrm{Id} \otimes \partial_i : \mathfrak{X}^{(p,q)} \to \mathfrak{X}^{(p,q+1)}, \qquad 0 \le i \le q+1 \tag{2.55}$$

$$\uparrow\sigma_j = \mathrm{Id} \otimes \sigma_j : \mathfrak{X}^{(p,q)} \to \mathfrak{X}^{(p,q-1)}, \qquad 0 \le j \le q-1 \tag{2.56}$$

$$\uparrow\tau = \mathrm{Id} \otimes \tau : \mathfrak{X}^{(p,q)} \to \mathfrak{X}^{(p,q)}, \tag{2.57}$$

where ∂_i, σ_i, and τ are defined by (2.44)-(2.47).

Proposition 2.16. *The bigraded module $\mathfrak{X}$ endowed with the above operators $(\overrightarrow{\partial}, \overrightarrow{\sigma}, \overrightarrow{\tau}, \uparrow\partial, \uparrow\sigma, \uparrow\tau)$ defines a bicocyclic module.*

Proof. The pth column of $\mathfrak{X}$ is identical to $N \otimes_{\mathcal{K}} D^{\otimes p} \otimes C^*_{\mathcal{H}}(C, M)$ and the vertical operators do not affect the horizontal part. So the columns are cocyclic modules.
To see that the rows are also cocyclic modules, one identifies the qth row with $C^*_{\mathcal{K}}(D, N \otimes M \otimes C^{\otimes q+1})$, where $\mathcal{K}$ acts on $N \otimes M \otimes C^{\otimes q+1}$ via its action on N, and coacts as follows. For $n \otimes m \otimes \tilde{c} \in N \otimes M \otimes C^{\otimes q+1}$ the coaction is

$$n \otimes m \otimes \tilde{c} \mapsto \tilde{c}_{<-1>} m_{<\overline{-1}>} \otimes n_{<\overline{0}>} \otimes m \otimes \tilde{c}_{<0>}.$$

Using the fact that N is a SAYD module over $\mathcal{K}$ and that $\mathcal{K}$ is commutative, one sees that $N \otimes M \otimes C^{\otimes q+1}$ is a right-left SAYD module over $\mathcal{K}$, hence the row forms a cocyclic module.
To finish the proof it remains to show that the vertical and horizontal operators commute among each other. We shall only check the nontrivial part, that is the commutation of $\uparrow\tau$ and $\overrightarrow{\tau}$. Using (2.38) one can write

$$\begin{aligned}
&\overrightarrow{\tau}\uparrow\tau(n \otimes \tilde{d} \otimes m \otimes \tilde{c}) = \\
&\overrightarrow{\tau}(n \otimes \tilde{d} \otimes m_{<\bar{0}>} \otimes c^1 \otimes \ldots \otimes c^q \otimes m_{<\overline{-1}>} c^0) = \\
&n_{<\bar{0}>} \otimes d^1 \otimes \ldots \otimes d^p \otimes n_{<\overline{-1}>} m_{<\overline{-1}><-1>} \tilde{c}_{<-1>} d^0 \otimes \\
&m_{<\bar{0}>} \otimes c^1{}_{<0>} \otimes \ldots \otimes c^q \otimes m_{<\overline{-1}><0>} c^0 = \\
&n_{<\bar{0}>} \otimes d^1 \otimes \ldots \otimes d^p \otimes \tilde{c}_{<-1>} n_{<\overline{-1}>} d^0 \otimes \\
&m_{<\overline{-1}>} \otimes c^1{}_{<0>} \otimes \ldots \otimes c^q{}_{<0>} m_{<\overline{-1}>} c^0{}_{<0>} = \\
&\uparrow\tau(n_{<\bar{0}>} \otimes d^1 \otimes \ldots \otimes d^p \otimes \tilde{c}_{<-1>} n_{<\overline{-1}>} d^0 \otimes m \otimes \tilde{c}_{<0>}) = \\
&\uparrow\tau\overrightarrow{\tau}(n \otimes \tilde{d} \otimes m \otimes \tilde{c}).
\end{aligned}$$

$\square$

We denote the diagonal of $\mathfrak{X}$ by $\mathfrak{Y}$; it is a cocyclic module whose degree n component is

$$N \otimes_{\mathcal{K}} \otimes D^{\otimes n} \otimes M \otimes_{\mathcal{H}} C^{\otimes n},$$

with the following operators:

$$d_i = \overrightarrow{\partial}_i \uparrow\partial_i, \qquad 0 \le i \le n+1, \tag{2.58}$$

$$s_j = \overrightarrow{\sigma}_j \uparrow\sigma_j, \qquad 0 \le j \le n-1, \tag{2.59}$$

$$t = \overrightarrow{\tau}\uparrow\tau \tag{2.60}$$

Next we identify the cocyclic module $C^*_{\mathcal{H}\rtimes\mathcal{K}}(C \rtimes D, M \otimes N)$ with $\mathfrak{Y}$.

Proposition 2.17. *The following map*

$$\Psi : C^*_{\mathcal{H}\rtimes\mathcal{K}}(C \rtimes D, M \otimes N) \to \mathfrak{Y}$$

is well-defined and defines an isomorphism of cocyclic modules:

$$\begin{aligned}
\Psi(m \otimes n \otimes c^0 \rtimes d^0 \otimes \ldots \otimes c^n \rtimes d^n) &= n \otimes d^0 \otimes c^1{}_{<-n>} d^1 \otimes \ldots \\
&\otimes c^1{}_{<-2>} \ldots c^{n-1}{}_{<-2>} d^{n-1} \otimes c^1{}_{<-1>} \cdots c^n{}_{<-1>} d^n \\
&\otimes m \otimes c^0 \otimes c^1{}_{<0>} \otimes \ldots \otimes c^n{}_{<0>}.
\end{aligned}$$

Proof. One can check that the following expression defines an inverse for Ψ:

$$\Psi^{-1}(n \otimes \tilde{d} \otimes m \otimes \tilde{c}) = m \otimes n \otimes c^0 \rtimes d^0 \otimes c^1{}_{<0>} \rtimes S(c^1{}_{<-1>})d^1 \otimes \dots$$
$$\dots \otimes c^n{}_{<0>} \rtimes S(c^1{}_{<-n>} \dots c^n{}_{<-1>})d^n.$$

First we show that Ψ is well-defined. Let $h \in \mathcal{H}$, $k \in \mathcal{K}$, $\widetilde{h \rtimes k} \in (\mathcal{H} \rtimes \mathcal{K})^{\otimes n+1}$, $m \in M$ and $n \in N$. One observes that

$$\Delta^{(n)}(h \rtimes k) =$$
$$h_{(1)} \rtimes h_{(2)}{}_{<-1>} \dots h_{(n+1)}{}_{<-n>} k_{(1)} \otimes h_{(2)}{}_{<0>} \rtimes h_{(3)}{}_{<-1>} \dots h_{(n+1)}{}_{<-n+1>} k_{(2)} \otimes \dots$$
$$\dots \otimes h_{(n)}{}_{<0>} \rtimes h_{<n+1>}{}_{<-1>} k_{(n)} \otimes h_{(n+1)}{}_{<0>} \rtimes k_{(n+1)}.$$

Then by using (2.39) we have:

$$\Psi(m \otimes n \otimes (h \rtimes k)(\widetilde{c \rtimes d})) = \Psi(m \otimes n \otimes \Delta^{(n)}(h \rtimes k)(\widetilde{c \rtimes d}) =$$
$$\Psi(m \otimes n \otimes h_{(1)} c^0 \rtimes h_{(2)}{}_{<-1>} \dots h_{(n+1)}{}_{<-n>} k_{(1)} d^0 \otimes h_{(2)}{}_{<0>} c^1 \rtimes h_{(3)}{}_{<-1>} \dots$$
$$h_{(n+1)}{}_{<-n+1>} k_{(2)} d^1 \otimes \dots \otimes h_{(n+1)}{}_{<0>} c^n \rtimes k_{(n+1)} d^n) =$$
$$n \otimes h_{(2)}{}_{<-n>} \dots h_{(n+1)}{}_{<-n>} k_{(1)} c^1{}_{<-n>} d^0 \otimes \dots$$
$$\dots \otimes h_{(2)}{}_{<-n>} \dots h_{(n+1)}{}_{<-n>} k_{(n+1)} c^1{}_{<-1>} \dots c^n{}_{<-1>} d^n \otimes$$
$$m \otimes h_{(1)} c^0 \otimes h_{(2)}{}_{<0>} c^1{}_{<0>} \otimes \dots \otimes h_{(n+1)}{}_{<0>} c^n{}_{<0>} =$$
$$n h_{(2)}{}_{<-1>} \dots h_{(n+1)}{}_{<-1>} \otimes k_{(1)} c^1{}_{<-n>} d^0 \otimes \dots \otimes k_{(n+1)} c^1{}_{<-1>} \dots c^n{}_{<-1>} d^n \otimes$$
$$m \otimes h_{(1)} c^0 \otimes h_{(2)}{}_{<0>} c^1{}_{<0>} \otimes \dots \otimes h_{(n+1)}{}_{<0>} c^n{}_{<0>} =$$
$$n \otimes k_{(1)} c^1{}_{<-n>} d^0 \otimes \dots \otimes k_{(n+1)} c^1{}_{<-1>} \dots c^n{}_{<-1>} d^n \otimes$$
$$m \otimes h_{(1)} c^0 \otimes h_{(2)} c^1{}_{<0>} \otimes \dots \otimes h_{(n+1)} c^n{}_{<0>} =$$
$$nk \otimes c^1{}_{<-n>} d^0 \otimes \dots \otimes c^1{}_{<-1>} \dots c^n{}_{<-1>} d^n \otimes$$
$$mh \otimes c^0 \otimes h_{(2)} c^1{}_{<0>} \otimes \dots \otimes c^n{}_{<0>} = \Psi(mh \otimes nk \otimes \widetilde{c \rtimes d}).$$

The next task is to show that Ψ is a cyclic map. To do this we should check that Ψ commutes with the faces, the degeneracies and the cyclic operator. We will just check the commutativity of Ψ with τ and ∂_0, the other verifications being completely similar.

Indeed, by using (2.39), and (2.38) one can write

$$\begin{array}{l}
\Psi(\tau(m \otimes n \otimes \widetilde{c \rtimes d})) = \\
\Psi(m_{<\bar{0}>} \otimes n_{<\bar{0}>} \otimes c^1 \rtimes d^1 \otimes \ldots \otimes c^n \rtimes d^n \otimes m_{<\overline{-1}>} c^0 \rtimes n_{<\overline{-1}>} d^0) = \\
n_{<\bar{0}>} \otimes d^1 \otimes {c^2}_{<-n>} d^2 \otimes \ldots \otimes {c^2}_{<-1>} \ldots {c^n}_{<-1>} m_{<\overline{-1}><-1>} {c^0}_{<-1>} d^0 \otimes \\
m_{<\bar{0}><0>} \otimes c^1 \otimes {c^2}_{<0>} \otimes \ldots \otimes {c^n}_{<0>} \otimes m_{<\overline{-1}><0>} {c^0}_{<0>} = \\
n_{<\bar{0}>} \otimes d^1 \otimes {c^2}_{<-n>} d^2 \otimes \ldots \otimes {c^2}_{<-1>} \ldots {c^n}_{<-1>} {c^0}_{<-1>} d^0 \otimes \\
m_{<\bar{0}>} \otimes c^1 \otimes {c^2}_{<0>} \otimes \ldots \otimes {c^n}_{<0>} \otimes m_{<\overline{-1}>} {c^0}_{<0>} = \\
n_{<\bar{0}>} {c^1}_{<-1>} \otimes d^1 \otimes {c^2}_{<-n>} d^2 \otimes \ldots \otimes {c^2}_{<-1>} \ldots {c^n}_{<-1>} {c^0}_{<-1>} d^0 \otimes \\
m_{<\bar{0}>} \otimes {c^1}_{<0>} \otimes {c^2}_{<0>} \otimes \ldots \otimes {c^n}_{<0>} \otimes m_{<\overline{-1}>} {c^0}_{<0>} = \\
n_{<\bar{0}>} \otimes {c^1}_{<-n-1>} d^1 \otimes {c^1}_{<-n>} {c^2}_{<-n>} d^2 \otimes \ldots \otimes {c^1}_{<-1>} {c^2}_{<-1>} \ldots {c^n}_{<-1>} {c^0}_{<-1>} d^0 \otimes \\
m_{<\bar{0}>} \otimes {c^1}_{<0>} \otimes {c^2}_{<0>} \otimes \ldots \otimes {c^n}_{<0>} \otimes m_{<\overline{-1}>} {c^0}_{<0>} = \\
t\Psi(m \otimes n \otimes \widetilde{c \rtimes d})).
\end{array}$$

Fnally, using once more (2.39) we check that $\Psi\partial_0 = d_0\Psi$, as follows:

$$\begin{array}{l}
\Psi\partial_0(m \otimes n \otimes \widetilde{c \rtimes d}) = \\
\Psi(m \otimes n \otimes {c^0}_{(1)} \rtimes {{c^0}_{(2)}}_{<-1>} {d^0}_{(1)} \otimes {{c^0}_{(2)}}_{<0>} \rtimes {d^0}_{(2)} \otimes c^1 \rtimes d^1 \otimes \ldots \otimes c^n \rtimes d^n) = \\
n \otimes {{c^0}_{(2)}}_{<-n-1>} {d^0}_{(1)} \otimes {{c^0}_{(2)}}_{<-n>} {d^0}_{(2)} \otimes {{c^0}_{(2)}}_{<-n+1>} {c^1}_{<-n+1>} d^1 \otimes \ldots \\
\ldots \otimes {{c^0}_{(2)}}_{<-1>} {c^1}_{<-1>} \ldots {c^n}_{<-1>} d^n \otimes {c^0}_{(1)} \otimes {{c^0}_{(2)}}_{<0>} \otimes \ldots \otimes {c^n}_{<0>} = \\
n{{c^0}_{(2)}}_{<-1>} \otimes {d^0}_{(1)} \otimes {d^0}_{(2)} \otimes {c^1}_{<-n+1>} d^1 \otimes \ldots \\
\ldots \otimes {c^1}_{<-1>} \ldots {c^n}_{<-1>} d^n \otimes {c^0}_{(1)} \otimes {{c^0}_{(2)}}_{<0>} \otimes \ldots \otimes {c^n}_{<0>} = \\
n \otimes {d^0}_{(1)} \otimes {d^0}_{(2)} \otimes {c^1}_{<-n+1>} d^1 \otimes \ldots \\
\ldots \otimes {c^1}_{<-1>} \ldots {c^n}_{<-1>} d^n \otimes {c^0}_{(1)} \otimes {c^0}_{(2)} \otimes {c^3}_{<0>} \otimes \ldots \otimes {c^n}_{<0>} = \\
d_0\Psi(m \otimes n \otimes \widetilde{c \rtimes d}).
\end{array}$$

$\square$

We have thus proved the following generalization of Theorem 2.11.

Theorem 2.18. *The mixed complexes* $(C^*_{\mathcal{H} \rtimes \mathcal{K}}(C \rtimes D, M \otimes N), b, B)$ *and* $(\mathrm{Tot}(\mathfrak{X}), b_T, B_T))$ *are quasi-isomorphic.*

3 A Cartan homotopy formula for Hopf cyclic cohomology

In this section we prove a homotopy formula for Hopf cyclic cohomology of coalgebras with coefficients in SAYD modules, which in conjunction with the spectral sequence of the previous section will allow us to compute the Hopf cyclic cohomology of Hopf algebras such as $\mathcal{H}_1^\dagger$ and $\mathcal{H}_1$. To put it into the proper perspective, we begin by recalling the cyclic cohomological version of the Cartan-type homotopy formula (cf. [4, Part II, Prop. 5], [21, §4.1]) for actions of derivations on algebras.

Let $\mathcal{A}$ be an algebra and let $D : \mathcal{A} \to \mathcal{A}$ be a derivation. Then D extends to a 'Lie derivative' operator, acting on the cochain complex of $\mathcal{A}$ by

$$\mathcal{L}_D f(a^0 \otimes \ldots \otimes a^n) = \sum_{i=0}^{n} f(a_0 \otimes \ldots \otimes a_{i-1} \otimes D(a_i) \otimes a_{i+1} \ldots \otimes a_n).$$

The 'contraction' by D is made from the operators

$$e_D : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A}) \quad \text{and} \quad E_D : C^n(\mathcal{A}) \to C^{n-1}(\mathcal{A})$$

defined by

$$e_D f(a_0 \otimes \ldots \otimes a_{n+1}) = (-1)^n f(D(a^n) a^0 \otimes \ldots \otimes a^{n-1}),$$

$$\begin{aligned} &E_D f(a_0 \otimes \ldots \otimes a_{n-1}) = \\ &\sum_{1 \le i \le j \le n} (-1)^{n(i+1)} f(1 \otimes a_i \otimes a_{i+1} \otimes \ldots \otimes a_{j-1} \otimes D(a_j) \otimes a_{j+1} \otimes \ldots \\ &\ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}) \end{aligned}$$

and the following relations are satisfied (cf. [21, §4.1]):

$$[e_D, b] = 0\,, \qquad [E_D, B] = 0\,,$$

$$[e_D, B] + [E_D, b] = \mathcal{L}_D\,.$$

As an example, we can take $\mathcal{A} = C_c^\infty(F^+M) \rtimes \Gamma$ as in §1.1, with $D = Y \in \mathcal{H}_1$ acting as a derivation on $\mathcal{A}$.

We can pullback the above operators on $C^n(\mathcal{H}_1)$ via the characteristic map

$$\chi : \mathcal{H}_1^{\otimes n} \to Hom(\mathcal{A}^{\otimes n+1}, \mathbb{C}),$$

$$\chi(h^1 \otimes \ldots \otimes h^n)(a_0 \otimes \ldots \otimes \alpha_n) = \tau(a_0 h^1(a_1) \ldots h^n(a_n))$$

and obtain operators $\mathcal{L}_Y^{tr}$, e_Y^{tr} and E_Y^{tr} respectively. The first one can be intrinsically expressed as follows.

Lemma 3.1. *In the above notation,*

$$\mathcal{L}_Y^{tr} = \mathrm{Id} - \mathrm{ad}\, Y : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}.$$

Proof. Recalling that the invariant trace τ satisfies $\tau(h(a)b) = \tau(aS_\delta(h)b)$, and $S_\delta(Y) = -Y + 1$, one obtains

$$\begin{aligned}
&\mathcal{L}_Y \chi(h^1 \otimes \ldots \otimes h^n)(a_0 \otimes \ldots \otimes a_n) = \\
&= \chi(h^1 \otimes \ldots \otimes h^n)(Y(a_0) \otimes a_1 \otimes \ldots \otimes a_n) + \\
&+ \sum_{i>0} \chi(h^1 \otimes \ldots \otimes h^n)(a_0 \otimes \ldots \otimes \otimes a_{i-1} \otimes Y(a_i) \otimes a_{i+1} \otimes \ldots \otimes a_n) \\
&= \tau(Y(a_0) h^1(a_1) \ldots h^n(a_n)) + \sum_{i>0} \tau(a_0 h^1(a_1) \ldots h^i(Y(a_i)) \ldots h^n(a_n)) \\
&= \tau(a_0 S_\delta(Y)(h^1(a_1) \ldots h^n(a_n)) + \sum_{i>0} \tau(a_0 h^1(a_1) \ldots h^i(Y(a_i)) \ldots h^n(a_n)) \\
&= \tau(a_0 h^1(a_1) \ldots h^n(a_n)) - \sum_{i>0} \tau(a_0 h^1(a_1) \ldots Y(h^i(a_i)) \ldots h^n(a_n)) + \\
&+ \sum_{i>0} \tau(a_0 h^1(a_1) \ldots h^i(Y(a_i)) \ldots h^n(a_n)) \\
&= \tau(a_0 h^1(a_1) \ldots h^n(a_n)) - \sum_{i>0} \tau(a_0 h^1(a_1) \ldots [Y, h^i](a_i)) \ldots h^n(a_n)).
\end{aligned}$$

□

Similarly, one can obtain intrinsic expressions for the pullbacks of the operators e_Y^{tr} and E_Y^{tr}.

Guided by this example, we proceed to find the analogous homotopy formula in the general context of Hopf cyclic cohomology of coalgebras with coefficients in SAYD modules.

Let $D : C \to C$ be an $\mathcal{H}$-linear coderivation, that is an $\mathcal{H}$-linear map such that

$$\Delta(D(c)) = D(c_{(1)}) \otimes c_{(2)} + c_{(1)} \otimes D(c_{(2)}).$$

We define the operators $\mathcal{L}_D : \mathcal{C}^n \to \mathcal{C}^n$, $e_D : \mathcal{C}^n \to \mathcal{C}^{n+1}$ and $E_D : \mathcal{C}^n \to \mathcal{C}^{n-1}$ on $\mathcal{C}^n$ as follows:

$$\mathcal{L}_D(m \otimes c^0 \otimes \ldots \otimes c^n) = \sum_{i=0}^{n} m \otimes c^0 \otimes \ldots \otimes D(c_i) \otimes \ldots \otimes c^n,$$

$$e_D(m \otimes c^0 \otimes \ldots \otimes c^n) = (-1)^n m_{<\overline{0}>} \otimes {c^0}_{(2)} \otimes c^1 \otimes \ldots \otimes c^n \otimes m_{<\overline{-1}>} D({c^0}_{(1)}),$$

$$E_D = \sum_{j=1}^{n} \sum_{i=1}^{j} E_D^{j,i} : \mathcal{C}^n \to \mathcal{C}^{n-1},$$

where

$$E_D^{j,i}(m \otimes c^0 \otimes \ldots \otimes c^n) =$$
$$= (-1)^{n(i+1)} \epsilon(c^0) m_{<\overline{0}>} \otimes c^{n-i+2} \otimes \ldots \otimes c^{n+1} \otimes m_{<\overline{-1}>} c_1 \otimes \ldots \otimes m_{<\overline{-(j-i)}>} c^{j-i} \otimes$$
$$\otimes m_{<\overline{-(j-i+1)}>} D(c^{j-i+1}) \otimes m_{<\overline{-(j-i+2)}>} c^{j-i+2} \otimes \ldots \otimes m_{<\overline{-(n-i+1)}>} c^{n-i+1}.$$

Let us also introduce the auxiliary operators $\psi_j : \mathcal{C}^n \to \mathcal{C}^n$,

$$\psi_j(m \otimes c^0 \otimes \ldots \otimes c^n) = m \otimes c^0 \otimes \ldots \otimes D(c_j) \otimes \ldots \otimes c^n.$$

Lemma 3.2. *The operators $\mathcal{L}_D$, e_D and E_D are well-defined and the following identities hold:*

$$\mathcal{L}_D = \sum_{j=0}^{n} \psi_j,$$
$$e_D = (-1)^n \tau \psi_0 \partial_0 = (-1)^n \psi_{n+1} \partial_{n+1}$$
$$E_D = \sum_{j=1}^{n} \sum_{i=1}^{j} (-1)^{n(i+1)} \psi_j \tau^{-i} \sigma_n \tau.$$

Proof. The operators $\mathcal{L}_D$, E_D, and e_D are well-defined as soon as ψ_js are shown to be well-defined. The latter property is easily checked, because D is $\mathcal{H}$-linear.

The first two identities being obviously true, we only check the last one:

$$\psi_j\tau^{-i}\sigma_n\tau(m\otimes c^0\otimes\ldots\otimes c^n)=\psi_j\tau^{-i}(\epsilon(c_0)m\otimes c^1\otimes\ldots\otimes c^n)$$
$$=\psi_j(\epsilon(c^0)m_{<\overline{0}>}\otimes S^{-1}(m_{<\overline{-1}>})c^{n-i+2}\otimes\ldots\otimes S^{-1}(m_{<\overline{-i}>})c^{n+1}\otimes c_1\otimes\ldots$$
$$\ldots\otimes c^{n-i+1}$$
$$=\epsilon(c^0)m_{<\overline{0}>}\otimes c^{n-i+2}\otimes\ldots\otimes c^{n+1}\otimes m_{<\overline{-1}>}c_1\otimes\ldots\otimes m_{<\overline{-(j-i)}>}c^{j-i}\otimes$$
$$\otimes\, m_{<\overline{-(j-i+1)}>}D(c^{j-i+1})\otimes m_{<\overline{-(j-i+2)}>}c^{j-i+2}\otimes\ldots\otimes m_{<\overline{-(n-i+1)}>}c^{n-i+1}.$$

□

Lemma 3.3. *The operator e_D and E_D commute, in the graded sense, with the Hochschild coboundary b and with the Connes boundary B respectively:*

$$[b,e_D]=0,\qquad [B,E_D]=0.$$

Proof. The commutation of E_D and B holds because of the fact that we are always working in the normalized subcomplex.
For e_D and b, one sees that $\partial_i e_D=e_D\partial_i$, for $0\le i\le n$, then by using the coderivation property of D one checks that $\partial_{n+1}e_D-\partial_{n+2}e_D=e_D\partial_{n+1}$. □

Lemma 3.4.

$$e_DB=-\sum_{k=1}^{n}(-1)^{(n+1)(n-k+2)}(-1)^kE_D^{n,n-k+1}\partial_k.$$

Proof.

$$e_DB=\sum_{i=0}^{n-1}(-1)^{n-1}(-1)^{ni}\psi_n\partial_n\tau^i\sigma_{n-1}\tau=\sum_{i=0}^{n-1}(-1)^{ni+n-1}\psi_n\tau\partial_0\tau^i\sigma_{n-1}\tau$$
$$=\sum_{i=0}^{n-1}(-1)^{ni+n-1}\psi_n\tau^i\partial_i\sigma_{n-1}\tau=\sum_{i=0}^{n-1}(-1)^{ni+n-1}\psi_n\tau^{i+1}\sigma_n\partial_i\tau$$
$$=\sum_{i=0}^{n-1}(-1)^{ni+n-1}\psi_n\tau^{i+1}\sigma_n\tau\partial_{i+1}=-\sum_{k=1}^{n}(-1)^{(n+1)(n-k+2)}(-1)^k\psi_n\tau^k\sigma_n\tau\partial_k$$
$$=-\sum_{k=1}^{n}(-1)^{(n+1)(n-k+2)}(-1)^kE_D^{n,n-k+1}\partial_k.$$

□

Lemma 3.5.

$$Be_D = \sum_{i=0}^{n} (-1)^{n+ni} \psi_{n-i} \tau^{i+1}.$$

Proof.

$$Be_D = \sum_{i=0}^{n} (-1)^{n+ni} \tau^i \sigma_n \tau \psi_{n+1} \partial_{n+1} = \sum_{i=0}^{n} (-1)^{n+ni} \tau^i \sigma_n \psi_n \tau \partial_{n+1}$$
$$= \sum_{i=0}^{n} (-1)^{n+ni} \tau^i \psi_n \sigma_n \partial_n \tau = \sum_{i=0}^{n} (-1)^{n+ni} \psi_{n-i} \tau^{i+1}$$

□

Lemma 3.6.

$$\mathcal{L}_D = Be_D + E_D \partial_0 + (-1)^{n+1} E_D \partial_{n+1}.$$

Proof. Using Lemma 3.5, one obtains

$$Be_D + E_D \partial_0 + (-1)^{n+1} E_D \partial_{n+1} = \sum_{i=0}^{n} (-1)^{n+ni} \psi_{n-i} \tau^{i+1} +$$
$$+ \sum_{j=1}^{n} \sum_{i=1}^{j} (-1)^{(n+1)} E_D^{j,i} \partial_{n+1} + E_D \partial_0 = (\psi_0 + \sum_{j=1}^{n} E_D^{j,1} \partial_{n+1}) +$$
$$+ (\sum_{i=0}^{n-1} (-1)^{n+ni} \psi_{n-i} \tau^{i+1} + \sum_{j=1}^{n} \sum_{i=2}^{j} (-1)^{n+1} E_D^{j,i} \partial_{n+1} + E_D \partial_0).$$

To finish the proof we just need to show that the result of the first parentheses is $\mathcal{L}_D$ and that of the second one vanishes. To this end, one checks that

$$E_D^{j,i} \partial_0 = (-1)^{(n+1)i} \psi_j \tau^{n+1-i},$$

and

$$E_D^{j,i} \partial_{n+1} = (-1)^{(n+1)(i+1)} \psi_j \tau^{n+2-i}.$$

□

Proposition 3.7.

$$[e_D + E_D, b + B] = \mathcal{L}_D.$$

Proof. From Lemma 3.3 we already know that $[E_D, B] = 0$ and $[e_D, b] = 0$. To complete the proof, after using Lemmas 3.4, 3.5, and 3.6, it suffices to show that:

$$\partial_k E^{j,i} = \begin{cases} E_D^{j,i}\partial_{k-i+1}, & k = i+1, \dots, j-1; \\ E_D^{j,i}\partial_{k-i+1}, & k = j+1 \dots n; \\ E^{j,i}\partial_{k-i+1} + E^{j+1,i}\partial_{k-i+1}, & k = j; \\ E^{j+1,i}\partial_{n-i+k}, & k = 1, \dots, i-1; \\ E^{j+1,i}\partial_1, & \text{k=i.} \end{cases}$$

We check the first case and leave the rest to the reader. Indeed,

$$\begin{aligned} &E^{j,i_D}\partial_{k-i+1} = \psi_j \tau^{-i}\sigma_n \tau \partial_{k-i+1} = \psi_j \tau^{-i}\sigma_n \partial_{k-i}\tau = \psi_j \partial_k \tau^i \sigma_{n-1}\tau \\ &= \partial_k \psi_j \tau^{-i}\sigma_{n-1}\tau = \partial_k E_D^{j,i}. \end{aligned}$$

□

Consider now a Hopf algebra $\mathcal{H}$ endowed with an MPI (δ, σ), and let $Z \in \mathcal{H}$ be a primitive element, i.e.,

$$\Delta(Z) \,=\, Z \otimes 1 + 1 \otimes Z \qquad \text{and} \qquad \epsilon(Z) \,=\, 0\,.$$

Then $D_Z : \mathcal{H} \to \mathcal{H}$ defined by

$$D_Z(h) = hZ$$

is an $\mathcal{H}$-linear coderivation, and we denote by $\mathcal{L}_Z$ its functorial extension to the tensor algebra of $\mathcal{H}$.

Lemma 3.8.

$$\mathcal{L}_Z = \delta(Z)\,\mathrm{Id} - \mathrm{ad}\, Z$$

Proof. Let $\Theta : C^n_H(\mathcal{H}, \ {}^{\sigma}\mathbb{C}_\delta) \to \mathcal{H}^{\otimes n}$ be the canonical isomorphism defined by the assignment

$$h^0 \otimes \dots \otimes h^n \mapsto S_\delta(h^0) \cdot h^1 \otimes \dots \otimes h^n\,,$$

with inverse

$$h^1 \otimes \dots \otimes h^n \mapsto 1 \otimes h^1 \otimes \dots \otimes h^n.$$

Since $S_\delta(Z) = \delta(Z) - Z$, one gets

$$\begin{aligned}
&\Theta\mathcal{L}_Z\Theta^{-1}(h^1 \otimes \ldots \otimes h^n) = \Theta\mathcal{L}_Z(1 \otimes h^1 \otimes \ldots \otimes h^n) = \\
&= \Theta(Z \otimes h^1 \otimes \ldots \otimes h^n) + \sum_{i+1}^{n} \Theta(1 \otimes h^1 \otimes \ldots \otimes h^i Z \otimes \ldots \otimes h^n) \\
&= \delta(Z) h^1 \otimes \ldots \otimes h^n \otimes - \sum_{i+1}^{n} (h^1 \otimes \ldots \otimes Z h^i \otimes \ldots \otimes h^n) + \\
&+ \sum_{i=1}^{n} h^1 \otimes \ldots \otimes h^i Z \otimes \ldots \otimes h^n = (\delta(Z)\,\mathrm{Id} - \mathrm{ad}_Z)(h^1 \otimes \ldots \otimes h^n).
\end{aligned}$$

□

To illustrate the application of this homotopy we consider the case of $\mathcal{H}_1$, with the primitive element $Y \in \mathcal{H}_1$ in the role of Z. First, from Lemma 3.8 and Proposition 3.7, one obtains:

Corollary 3.9. (The Cartan Homotopy Formula for $\mathcal{H}_1$)

$$\mathrm{Id} - \mathrm{ad}\, Y \;=\; [E_Y + e_Y, B + b]\,.$$

We now look at the concrete way in which $\mathrm{ad}\, Y$ acts on $\mathcal{H}_1$. Since

$$\mathrm{ad}\, Y(Y) = 0\,, \quad \mathrm{ad}\, Y(X) = X \quad \text{and} \quad \mathrm{ad}\, Y(\delta_k) = k\delta_k\,, \quad \forall\, k \geq 1\,,$$

one sees that, for a typical basis element $Y^p X^q \delta_1^{r_1} \ldots \delta_m^{r_m}$ of $\mathcal{H}_1$,

$$\mathrm{ad}\, Y(Y^p X^q \delta_1^{r_1} \ldots \delta_m^{r_m}) = (q + r_1 \cdots + r_m) Y^p X^q \delta_1^{r_1} \ldots \delta_m^{r_m}.$$

Thus, $\mathrm{ad}\, Y$ is diagonalizable, acting as a grading operator, with non-negative integer weights. In particular, the weight 1 eigenspace is

$$\mathcal{H}_1[1] \;=\; \mathbb{C}[Y]X \otimes \mathbb{C}[Y]\delta_1.$$

The above grading extends to the cocyclic module $\mathcal{H}_{1\natural}$ and gives a corresponding decomposition

$$\mathcal{H}_{1\natural} = \bigoplus_{k \geq 0} \mathcal{H}_{1\natural}[k],$$

where $\mathcal{H}_{1\natural}[k] = \bigoplus_{n\geq 0}\{\tilde{h} \in \mathcal{H}_1^{\otimes n} \mid |\tilde{h}| = k\}$, with $|h^1\otimes\ldots\otimes h^n| = |h^1|+\ldots+|h^n|$ denoting the total weight.
Since the operators defining the cocyclic structure of $\mathcal{H}_{1\natural}$ obviously preserve the weight, the above decomposition is actually a decomposition of cocyclic modules. Then Corollary 3.9 has in turn the following implications.

Corollary 3.10.

$$HP^i_{(\delta,1)}(\mathcal{H}_{1\natural}[1]) = HP^i(\mathcal{H}_1) \quad \text{and} \quad HP^i_{(\delta,1)}(\mathcal{H}_{1\natural}[k]) = 0, \quad \forall\, k \neq 1.$$

4 Hopf cyclic cohomology of Hopf algebras in codimension 1

4.1 Hopf cyclic cohomology of $\mathcal{H}_1$ and $\mathcal{H}_{1s}$

By its very construction (cf. [7]), the Hopf algebra $\mathcal{H}_1$ is a bicrossed product of two Hopf subalgebras. We shall use this feature to construct a spectral sequence converging to the Hochschild cohomology of $\mathcal{H}_1$. This spectral sequence is further graded by the weight, and its weight 1 component is extremely simple. This will allow us, first to explicitly compute the weight 1 component of the Hochschild cohomology and then, in conjunction with Corollary 3.10, to finish off the calculation of the periodic Hopf cyclic cohomology of $\mathcal{H}_1$.

To begin with, we shall make the bicrossed product construction completely explicit (cf. [7], also [20]). Let $\mathcal{U} = \mathcal{U}(\mathfrak{g}_-)$ denote the universal enveloping algebra of the Lie algebra $\mathfrak{g}_-$ of the '$ax+b$' group of affine diffeomorphisms of the real line. The Lie algebra $\mathfrak{g}_-$ has generators X, Y with the bracket

$$[Y, X] = X. \tag{4.1}$$

Let $\mathcal{F} = \mathcal{F}(G_+)$ be the (commutative) Hopf subalgebra of $\mathcal{H}_1$ generated by $\{\delta_k\,;\, k \in \mathbb{N}\}$, which we may view as algebra of regular functions on the pronilpotent Lie group G_+ consisting of the diffeomorphisms $\psi \in \mathrm{Diff}(\mathbb{R})$ that satisfy $\psi(0) = 0$ and $\psi'(0) = 1$.

The Hopf algebra $\mathcal{U}$ acts from the right on the Hopf algebra $\mathcal{F}$ via

$$\delta_k \triangleleft X = -\delta_{k+1}, \quad \delta_k \triangleleft Y = -k\delta_k,$$

making $\mathcal{F}$ a right $\mathcal{U}$ -module algebra.
On the other hand, the Hopf algebra $\mathcal{F}$ coacts from the left on $\mathcal{U}$ via the coaction $\rho : \mathcal{U} \to \mathcal{F} \otimes \mathcal{U}$ defined by

$$\rho(X) = 1 \otimes X + \delta_1 \otimes Y, \tag{4.2}$$
$$\rho(Y) = 1 \otimes Y, \tag{4.3}$$
$$\rho(1) = 1 \otimes 1, \tag{4.4}$$

and extends to the basis of $\mathcal{U}$ via

$$\rho(u^1u^2) = (u^1{}_{<-1>} \triangleleft u^2{}_{(1)})u^2{}_{(2)}{}_{<-1>} \otimes u^1{}_{<0>}u^2{}_{(2)}{}_{<0>}. \tag{4.5}$$

With respect to this coaction $\mathcal{U}$ is a left $\mathcal{F}$-comodule coalgebra, Moreover, it is not difficult to check that the above action and coaction satisfy the axioms [22, page 232] required to form the bicrossed product. The resulting Hopf algebra coincides as coalgebra with the cocrossed product coalgebra $\mathcal{U} \ltimes \mathcal{F}$, and as algebra with the crossed product $\mathcal{U} \rtimes \mathcal{F}$. We identify X , Y and $\delta_k \in \mathcal{H}_1$ with $X \rtimes 1$, $Y \rtimes 1$ and $1 \rtimes \delta_k$ in $\mathcal{U} \blacktriangleright\!\!\triangleleft \mathcal{F}$, respectively.

Lemma 4.1. *$\mathcal{H}_1$ and $\mathcal{U} \blacktriangleright\!\!\triangleleft \mathcal{F}$ are isomorphic as Hopf algebras.*

Proof. Using the fact that the algebra structure of $U \blacktriangleright\!\!\triangleleft F$ is the crossed product algebra, we first show that $X \rtimes 1$, $Y \rtimes 1$ and $1 \rtimes \delta_k$ satisfy the identities (1.9). Indeed,

$$[Y \rtimes 1, X \rtimes 1] = YX \rtimes 1 - XY \rtimes 1 = X \rtimes 1,$$

$$[X \rtimes 1, 1 \rtimes \delta_k] = (X \rtimes 1)(1 \rtimes \delta_k) - (1 \rtimes \delta_k)(X \rtimes 1) =$$
$$X \rtimes \delta_k - X \rtimes \delta_k - 1 \rtimes \delta_k \triangleleft X = 1 \rtimes \delta_{k+1},$$

$$[Y \rtimes 1, 1 \rtimes \delta_k] = (Y \rtimes 1)(1 \rtimes \delta_k) - (1 \rtimes \delta_k)(Y \rtimes 1) =$$
$$Y \rtimes \delta_k - Y \rtimes \delta_k - 1 \rtimes \delta_k \triangleleft Y = 1 \rtimes k\delta_k,$$

$$[1 \rtimes \delta_j, 1 \rtimes \delta_k] = 1 \rtimes \delta_j\delta_k - 1 \rtimes \delta_k\delta_j = 0.$$

Now we use the coalgebra structure of $\mathcal{U} \blacktriangleright\!\!\vartriangleleft \mathcal{F}$ which is of a cocrossed product coalgebra to check the identities (1.10)–(1.13):

$$\Delta(X \rtimes 1) = X_{(1)} \rtimes X_{(2)_{<-1>}} \otimes X_{(2)_{<0>}} \rtimes 1 =$$
$$X \rtimes 1 \otimes 1 \rtimes 1 + 1 \rtimes X_{<-1>} \rtimes X_{<0>} \rtimes 1 =$$
$$X \rtimes 1 \otimes 1 \rtimes 1 + 1 \rtimes X \otimes 1 \rtimes 1 + 1 \rtimes \delta_i \otimes Y \rtimes 1,$$

$$\Delta(Y \rtimes 1) = Y_{(1)} \rtimes Y_{(2)_{<-1>}} \otimes Y_{(2)_{<0>}} \rtimes 1 =$$
$$Y \rtimes 1 \otimes 1 \rtimes 1 + 1 \rtimes Y_{<-1>} \rtimes Y_{<0>} \rtimes 1 =$$
$$Y \rtimes 1 \otimes 1 \rtimes 1 + 1 \rtimes Y \otimes 1 \rtimes 1,$$

$$\Delta(1 \rtimes \delta_1) = 1 \rtimes \delta_{1(1)} \otimes 1 \rtimes \delta_{1(2)} = 1 \rtimes \delta_1 \otimes 1 \rtimes 1 + 1 \rtimes 1 \otimes 1 \rtimes \delta_1.$$

Finally, using that the counit of $\mathcal{U} \blacktriangleright\!\!\vartriangleleft \mathcal{F}$ is

$$\epsilon(h \rtimes f) = \epsilon(h)\epsilon(f)\,,$$

one easily verifies that

$$\epsilon(X \rtimes 1) = \epsilon(Y \rtimes 1) = \epsilon(1 \rtimes \delta_k) = 0.$$

$\square$

The maps Ψ and Ψ^{-1} defined in §2.2, although not shown to be cyclic (comp. Proposition 2.10) in the case at hand , are easily seen to preserve the cosimplicial structure. As a matter of fact the latter property holds for any Hopf algebra whose underlying coalgebra is a cocrossed product. Thus, at least at the cosimplicial level, the diagonal subcomplex is still quasi-isomorphic to the Hochschild complex associated to the bicosimplicial complex whose component in bidegree (p,q) is $\mathcal{F}^{\otimes p} \otimes \mathcal{U}^{\otimes q}$. More precisely, we are interested in the cohomology of the total complex of the following bicomplex:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\mathcal{U}^{\otimes 2} \arrow[u, "\uparrow b"] \arrow[r, "\overrightarrow{b}"] & \mathcal{F} \otimes \mathcal{U}^{\otimes 2} \arrow[u, "\uparrow b"] \arrow[r, "\overrightarrow{b}"] & \mathcal{F}^{\otimes 2} \otimes \mathcal{U}^{\otimes 2} \arrow[u, "\uparrow b"] \arrow[r, "\overrightarrow{b}"] & \cdots \\
\mathcal{U} \arrow[u, "\uparrow b"] \arrow[r, "\overrightarrow{b}"] & \mathcal{F} \otimes \mathcal{U} \arrow[u, "\uparrow b"] \arrow[r, "\overrightarrow{b}"] & \mathcal{F}^{\otimes 2} \otimes \mathcal{U} \arrow[u, "\uparrow b"] \arrow[r, "\overrightarrow{b}"] & \cdots \\
\mathbb{C} \arrow[u, "\uparrow b"] \arrow[r, "\overrightarrow{b}"] & \mathcal{F} \arrow[u, "\uparrow b"] \arrow[r, "\overrightarrow{b}"] & \mathcal{F}^{\otimes 2} \arrow[u, "\uparrow b"] \arrow[r, "\overrightarrow{b}"] & \cdots
\end{tikzcd}
$$

In order to evaluate it, we define a filtration on the total complex as follows:

$$F_i = \bigoplus_{p \geq i} \bigoplus_{q \geq 0} \mathcal{F}^{\otimes p} \otimes \mathcal{U}^{\otimes q} .$$

Lemma 4.2. *The E_1 term of the above spectral sequence is as follows:*

$$
\begin{array}{rcl}
E_1^{p,0} & = & \mathcal{F}^{\otimes p} , \qquad E_1^{p,1} \cong (\mathcal{F}^{\otimes p} \otimes \mathbb{C}X) \otimes (\mathcal{F}^{\otimes p} \otimes \mathbb{C}Y) , \\
E_1^{p,2} & \cong & \mathcal{F}^{\otimes p} \otimes \mathbb{C}X \wedge Y , \qquad E_1^{p,q} = 0 , \quad \forall\, q \geq 3 .
\end{array}
$$

Proof. The term E_1 is the cohomology of the columns with respect to the boundary $\uparrow b$ defined in (2.26). In other words, $E_1^{p,q} = H^q(\mathcal{U}, \mathcal{F}^{\otimes p})$ and since $\mathcal{U}$ coacts trivially on $\mathcal{F}^{\otimes p}$, one gets

$$E_1^{p,q} \;=\; H^q(\mathcal{U}, \mathbb{C}) \otimes \mathcal{F}^{\otimes p} .$$

It is well-known that (cf. e.g. [19, 7]) the antisymmetrisation map induces an isomorphism between coalgebra Hochschild cohomology of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ and its exterior algebra $\wedge^* \mathfrak{g}$. Accordingly, E_1 is given by:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
0 & 0 & 0 & \\
\wedge^2\mathfrak{g} \arrow[u] & \mathcal{F}\otimes\wedge^2\mathfrak{g} \arrow[u] & \mathcal{F}^{\otimes 2}\otimes\wedge^2\mathfrak{g} \arrow[u] & \cdots \\
\mathfrak{g} \arrow[u, "0"] & \mathcal{F}\otimes\mathfrak{g} \arrow[u, "0"] & \mathcal{F}^{\otimes 2}\otimes\mathfrak{g} \arrow[u, "0"] & \cdots \\
\mathbb{C} \arrow[u, "0"] & \mathcal{F} \arrow[u, "0"] & \mathcal{F}^{\otimes 2} \arrow[u, "0"] & \cdots
\end{tikzcd}
$$

$\square$

Since all maps are weight preserving our spectral sequence is graded,

$$E_r = \bigoplus_{k\geq 0} E_r[k], \quad E_r[k] = \{\tilde{f}\otimes\tilde{u}\in E_r \mid \quad |\tilde{f}|+|\tilde{u}| = k\}.$$

Taking into consideration that we are working in the normalized complex, one sees that $E_1[1]$ reduces to

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & 0 & 0 & 0 & \cdots \\
 & \mathbb{C}X\wedge Y \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \cdots \\
E_1[1] & \mathbb{C}X \arrow[u, "0"] & \mathbb{C}\delta_1\otimes\mathbb{C}Y \arrow[u] & 0 \arrow[u] & \cdots \\
 & 0 \arrow[u] & \mathbb{C}\delta_1 \arrow[u, "0"] & 0 \arrow[u] & \cdots
\end{tikzcd}
$$

and it then easily follows that $E_2[1]$ looks as follows :

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & 0 & 0 & 0 & \cdots \\
 & \mathbb{C}X \wedge Y \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
E_2[1] & \mathbb{C}X \arrow[r, "\vec{b}"] & \mathbb{C}\delta_1 \otimes \mathbb{C}Y \arrow[r] & 0 \arrow[r] & \cdots \\
 & 0 \arrow[r] & \mathbb{C}\delta_1 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

Furthermore, since $\vec{b}(X) = \delta_1 \otimes Y$, the next stage simplifies it further to only two nontrivial classes:

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
 & 0 \arrow[rrd] & 0 \arrow[rrd] & 0 & 0 & \cdots \\
 & \mathbb{C}X \wedge Y \arrow[rrd] & 0 \arrow[rrd] & 0 & 0 & \cdots \\
E_3[1] & 0 \arrow[rrd] & 0 \arrow[rrd] & 0 & 0 & \cdots \\
 & 0 & \mathbb{C}\delta_1 & 0 & 0 & \cdots
\end{tikzcd}
$$

Proposition 4.3. *The weight* 1 *component of the Hochschild cohomology of* $H^*(\mathcal{H}_1)$ *is generated by the classes* $[\delta_1]$ *and* $[X \otimes Y - Y \otimes X - Y\delta_1 \otimes Y]$.

Proof. From the above picture, one sees that in order to prove the statement it suffices to convert the nontrivial weight 1 classes $[\delta_1] \in E_2^{1,0}$ and $X \wedge Y \in E_2^{0,2}$ into Hochschild classes of $\mathcal{H}_1$. This is a two-step process. First one needs to pass from the total complex to the diagonal of the bicosimplicial complex $\mathcal{F}^* \otimes \mathcal{U}^*$, via the Alexander-Whitney map (2.30). The second move

is the return to the Hochschild complex of $\mathcal{H}_1$, via the map Ψ^{-1} defined in Proposition 2.10.
The result of performing the first move is the following:

$$\begin{aligned}
&AW(X \otimes Y) = \vec{\partial}_2 \vec{\partial}_1 (X \otimes Y) = \vec{\partial}_2 (X_{(-1)} Y_{(-1)} \otimes X_{<0>} \otimes Y_{<0>}) = \\
&\vec{\partial}_2 (1 \otimes X \otimes Y) + \vec{\partial}_2 (\delta_1 \otimes Y \otimes Y) = \\
&1 \otimes X_{<-1>} Y_{<-1>} \otimes X_{<0>} \otimes Y_{<0>} + \delta_1 \otimes Y_{<-1>} Y_{<-1>} \otimes Y_{<0>} \otimes Y_{<0>} = \\
&1 \otimes \delta_1 \otimes Y \otimes Y + 1 \otimes 1 \otimes X \otimes Y + \delta_1 \otimes 1 \otimes Y \otimes Y,
\end{aligned}$$

and similarly

$$AW(Y \otimes X) = \delta_1 \otimes 1 \otimes Y \otimes Y + 1 \otimes \delta_1 \otimes Y \otimes Y + 1 \otimes 1 \otimes Y \otimes X.$$

This implies that

$$AW(X \wedge Y) = 1 \otimes 1 \otimes X \otimes Y - 1 \otimes 1 \otimes Y \otimes X.$$

On the other hand

$$\Psi^{-1}(1 \otimes 1 \otimes X \otimes Y) = X \rtimes 1 \otimes Y \rtimes 1 - Y \rtimes \delta_1 \otimes Y \rtimes 1 - Y \rtimes 1 \otimes Y \rtimes \delta_1,$$

$$\Psi^{-1}(1 \otimes 1 \otimes Y \otimes X) = Y \rtimes 1 \otimes X \rtimes 1 - Y \rtimes 1 \otimes Y \rtimes \delta_1\,,$$

and after the usual identification we obtain

$$\Psi^{-1}(1 \otimes 1 \otimes X \wedge Y) = X \otimes Y - Y \otimes X - Y\delta_1 \otimes Y.$$

In the case of $[\delta_1] \in E_2^{1,0}$ one has $AW(\delta_1) = -\delta_1 \otimes 1$, and hence $\Psi^{-1}(\delta_1 \otimes 1) = 1 \rtimes \delta_1$, identified to δ_1. $\square$

We are now in a position to give an intrinsic proof of one of the key results in [7], without appealing to Gelfand-Fuks cohomology.

Theorem 4.4. *The periodic cyclic cohomology of the Hopf algebra $\mathcal{H}_1$ with respect to the MPI $(\delta, 1)$ is the following:*

$$HP^{\text{odd}}_{(\delta,1)}(\mathcal{H}_1) = \mathbb{C} \cdot [\delta_1]\,, \qquad HP^{\text{ev}}_{(\delta,1)}(\mathcal{H}_1) = \mathbb{C} \cdot [X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y]\,.$$

Proof. From Corollary 3.10 we know that $HP^*_{(\delta,\sigma)}(\mathcal{H}_1)$ can be computed solely from the weight 1 mixed complex $(\mathcal{H}_{1\natural}[1], b, B)$. By the above proposition the Hochschild cohomology of the latter complex is generated by the Hochschild classes of δ_1 and $X \otimes Y - Y \otimes X - Y\delta_1 \otimes Y$. As both of them are cyclic, the Connes boundary B is zero on the Hochschild cohomology.
To obtain the stated form of the result it remains to note that the classes $[X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y]$ and $[X \otimes Y - Y \otimes X - Y\delta_1 \otimes Y]$ coincide, since

$$Y\delta_1 \otimes Y - \delta_1 Y \otimes Y \;=\; \delta_1 \otimes Y \;=\; b(X) \;=\; (b+B)(X)\,.$$

$\square$

A similar result holds for the quotient Hopf algebra $\mathcal{H}_{1s} = \mathcal{H}_1/\mathcal{S}$, where $\mathcal{S}$ is the ideal generated by the 'Schwarzian' $\delta_2' = \delta_2 - \frac{1}{2}\delta_1^2$. The Hopf algebra $\mathcal{H}_{1s}$ plays the crucial role in the extension of the Rankin-Cohen brackets to modular Hecke algebras (cf. [11]).

Evidently, as an algebra, $\mathcal{H}_{1s}$ is generated by (the classes *mod* $\mathcal{S}$ of) X, Y and $Z{=}\delta_1$, subject to the relations

$$[Y,X] = X, \quad [Y,Z] = Z, \quad [X,Z] = \frac{1}{2}Z^2;$$

its coalgebra structure is defined by

$$\begin{aligned}
&\Delta(X) = 1 \otimes X + X \otimes 1 + Z \otimes Y,\\
&\Delta(Y) = Y \otimes 1 + 1 \otimes Y,\\
&\Delta(Z) = Z \otimes 1 + 1 \otimes Z,\\
&\epsilon(X) = \epsilon(Y) = \epsilon(Z) = 0\,,
\end{aligned}$$

and the antipode is defined by

$$S(X) = -X + ZY, \quad S(Y) = -Y, \quad S(Z) = -Z.$$

Theorem 4.5. *The periodic cyclic cohomology of the Hopf algebra $\mathcal{H}_{1s}$ with respect to the MPI $(\delta, 1)$ is the following:*

$$HP^{\mathrm{odd}}_{(\delta,1)}(\mathcal{H}_{1s}) = \mathbb{C}\cdot[Z]\,, \qquad HP^{\mathrm{ev}}_{(\delta,1)}(\mathcal{H}_{1s}) = \mathbb{C}\cdot[X \otimes Y - Y \otimes X - ZY \otimes Y]\,.$$

Proof. We first reconstruct $\mathcal{H}_{1s}$ as a bicrossed product Hopf algebra, emulating Lemma 4.1. Indeed, the algebra $\mathcal{U} = \mathcal{U}(\mathfrak{g}_-)$ acts from the right on the polynomial algebra $\mathbb{C}[Z]$ via

$$Z \triangleleft X = -\frac{1}{2}Z^2, \quad Z \triangleleft Y = -Z\,,$$

and turns it into a right $\mathcal{U}$-module algebra. On the other hand, $\mathbb{C}[Z]$, viewed as the free Hopf algebra generated by the primitive element Z, coacts on $\mathcal{U}$ via $\rho : \mathcal{U} \to \mathbb{C}[Z] \otimes \mathcal{U}$ defined by

$$\rho(X) = 1 \otimes X + Z \otimes Y, \quad \rho(Y) = 1 \otimes Y\,,$$

Endowed with this coaction $\mathcal{U}$ is a $\mathbb{C}[Z]$-comodule coalgebra. Moreover, with respect to the above action and coaction, $\mathcal{U}$ and $\mathbb{C}[Z]$ form a matched pair of Hopf algebras (cf. [22, page 322]). One can then form the bicrossed product Hopf algebra $\mathcal{U} \blacktriangleright\!\!\triangleleft\, \mathbb{C}[Z]$, which is clearly isomorphic to $\mathcal{H}_{1s}$, via the isomorphism identifying $X \rtimes 1$, $Y \rtimes 1$, and $1 \rtimes Z$ with X, Y, and Z respectively.

To obtain the periodic cohomology $HP^*_{(\delta,1)}(\mathcal{H}_{1s})$, it suffices to repeat verbatim the proof of Theorem 4.4, after replacing $\mathcal{F}$ by $\mathbb{C}[Z]$. ∎

4.2 Hopf cyclic cohomology of $\mathcal{H}_1^\dagger$

In this section we compute the periodic cyclic cohomology of the Hopf algebra $\mathcal{H}_1^\dagger$ defined in §1.2, with coefficients in $\mathcal{K} = \mathbb{C}[\sigma, \sigma^{-1}]$ viewed as an SAYD module.

In order to make the method of §2 applicable, we shall represent $\mathcal{H}_1^\dagger$ as a cocrossed product coalgebra. To this end, we define a coaction of $\mathcal{K}$ on $\mathcal{H}_1$,

$$\rho : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}, \quad \rho(h) = \sigma^{|h|} \otimes h. \tag{4.6}$$

Lemma 4.6. *The map $\rho : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}$ defines a coaction, which makes $\mathcal{H}_1$ a $\mathcal{K}$-comodule Hopf algebra.*

Proof. The first claim is obvious. Next, $\mathcal{H}_1$ is a $\mathcal{K}$-comodule algebra because $\rho(1) = 1 \otimes 1$ and

$$\rho(hg) = \sigma^{|hg| \otimes hg} = \sigma^{|h|+|g|} \otimes hg = (\sigma^{|h|} \otimes h)(\sigma^{|g|} \otimes g) = \rho(h)\rho(g).$$

To prove that it is also a comodule coalgebra, we note that

$$h_{<-1>} \otimes \epsilon(h_{<0>}) = \sigma^{|h|} \otimes \epsilon(h) = 1 \otimes \epsilon(h),$$

where in the latter equality we used the fact that $\epsilon(h) = 0$ if $|h| > 0$.
Finally, the coaction and the coproduct do commute:

$$\begin{aligned}
&h_{(1)}{}_{<-1>} h_{(2)}{}_{<-1>} \otimes h_{(1)}{}_{<0>} \otimes h_{(2)}{}_{<0>} = \sigma^{|h_{(1)}|} \sigma^{|h_{(2)}|} \otimes h_{(1)} \otimes h_{(2)} \\
&= \sigma^{|h_{(1)}|+|h_{(2)}|} \otimes h_{(1)} \otimes h_{(2)} = \sigma^{|h|} \otimes h_{(1)} \otimes h_{(2)} = h_{<-1>} \otimes h_{<0>}{}_{(1)} \otimes h_{<0>}{}_{(2)}.
\end{aligned}$$

$\square$

Since $\mathcal{H}_1$, $\mathcal{K}$ and ρ satisfy the condition of Lemma 2.2, one can form the Hopf algebra $\mathcal{H}_1 \rtimes \mathcal{K}$.

Lemma 4.7. *The Hopf algebras $\mathcal{H}_1^\dagger$ and $\mathcal{H}_1 \rtimes \mathcal{K}$ are isomorphic.*

Proof. The map $\Phi : \mathcal{H}_1^\dagger \to \mathcal{H}_1 \rtimes \mathcal{K}$, defined on monomials by $\Phi(\sigma^m h) = h \rtimes \sigma^m$, extends by multiplicativity to an algebra map. We check that it commutes with coproducts and counits. Indeed,

$$\begin{aligned}
&\Delta_{\mathcal{H}_1 \rtimes \mathcal{K}}(\Psi(\sigma^m h)) = \Delta_{\mathcal{H}_1 \rtimes \mathcal{K}}(h \rtimes \sigma^m) = \\
&= h_{(1)} \rtimes h_{(2)}{}_{<-1>} \sigma^m{}_{(1)} \otimes h_{(2)}{}_{<0>} \rtimes \sigma^m{}_{(2)} \\
&= h_{(1)} \rtimes \sigma^{|h_{(2)}|+m} \otimes h_{(2)} \rtimes \sigma^m = \\
&= \Psi \otimes \Psi(\sigma^{m+|h_{(2)}|} h_{(1)} \otimes \sigma^m h_{(2)}) = \Psi \otimes \Psi(\Delta_{\mathcal{H}_1^\dagger}(\sigma^m h))
\end{aligned}$$

and

$$\epsilon_{\mathcal{H}_1 \rtimes \mathcal{K}}(\Psi(\sigma^m h)) = \epsilon_{\mathcal{H}_1 \rtimes \mathcal{K}}(h \rtimes \sigma^m) = \epsilon_{\mathcal{H}_1}(h)\epsilon_{\mathcal{K}}(\sigma^m) = \epsilon_{\mathcal{H}_1^\dagger}(h).$$

$\square$

We recall that $(\delta, 1)$ is a MPI for $\mathcal{H}_1$, with $\delta(X) = \delta(\delta_k) = 0$ and $\delta(Y) = 1$.
On the other hand, the MPIs for $\mathcal{K}$ fall into two classes: those of the form $(\beta, 1)$, where β is any character of $\mathcal{K}$, and those of the form $(\beta_{2k\pi i/n}, \sigma^n)$, $k \in \mathbb{Z}$, $0 \neq n \in \mathbb{Z}$, where $\beta_z(\sigma) = e^z$, $z \in \mathbb{C}$.
Among the MPIs listed above only those of the form (ϵ, σ^k), $k \in \mathbb{Z}$, for $\mathcal{K}$, together with $(\delta, 1)$, for $\mathcal{H}_1$, satisfy the conditions of Proposition 2.7, and thus combine to give $(\delta \otimes \epsilon, 1 \otimes \sigma^k)$, $k \in \mathbb{Z}$, MPIs for $\mathcal{H}_1^\dagger$.

Let us compute the Hopf cyclic cohomology of $\mathcal{H}_1^\dagger$ with coefficients in the MPI (δ, σ^k), where we identify $\delta \equiv \delta \otimes \epsilon$ and $\sigma^k \equiv 1 \rtimes \sigma^k$.

Since by Theorem 2.11 the mixed complex $(\mathcal{H}_1^\dagger, b, B)$ is quasi-isomorphic to $(\mathrm{Tot}(\mathfrak{C}), b_T, B_T)$, we start by computing the Hochschild cohomology of $\mathrm{Tot}(\mathfrak{C})$. To this end, we introduce the following filtration on $\mathrm{Tot}(\mathfrak{C})$:

$$F^n = \bigoplus_{q \geq n} \bigoplus_{p \geq 0} \mathcal{K}^{\otimes p} \otimes \mathcal{H}_1^{\otimes q}.$$

Thus, $E_1^{p,q}$ is the pth cohomology of the qth row in the diagram below:

$$E_0 \qquad \begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\mathcal{H}_1^{\otimes 2} & \xrightarrow{\vec{b}} & \mathcal{K} \otimes \mathcal{H}_1^{\otimes 2} & \xrightarrow{\vec{b}} & \mathcal{K}^{\otimes 2} \otimes \mathcal{H}_1^{\otimes 2} & \xrightarrow{\vec{b}} & \cdots \\
\mathcal{H}_1 & \xrightarrow{\vec{b}} & \mathcal{K} \otimes \mathcal{H}_1 & \xrightarrow{\vec{b}} & \mathcal{K}^{\otimes 2} \otimes \mathcal{H}_1 & \xrightarrow{\vec{b}} & \cdots \\
\mathbb{C} & \xrightarrow{\vec{b}} & \mathcal{K} & \xrightarrow{\vec{b}} & \mathcal{K}^{\otimes 2} & \xrightarrow{\vec{b}} & \cdots
\end{array}$$

This cohomology is precisely the Hochschild cohomology of $\mathcal{K}$ with coefficients in $\mathcal{H}_1^{\otimes q}$, viewed as a left $\mathcal{K}$-comodule via the coaction

$$\tilde{\rho} : \mathcal{H}_1^q \to \mathcal{K} \otimes \mathcal{H}_1^{\otimes q}, \quad \tilde{\rho}(\tilde{h}) = \sigma^{k+|\tilde{h}|} \otimes \tilde{h}.$$

To compute it, we use the free $\mathcal{K}$-comodule resolution for the trivial $\mathcal{K}$ comodule $\mathbb{C}$ given in [13], namely:

$$0 \to \mathbb{C} \xrightarrow{\eta} \mathbb{C}[\sigma, \sigma^{-1}] \xrightarrow{\theta} \mathbb{C}[\sigma, \sigma^{-1}] \xrightarrow{\gamma} \mathbb{C}[\sigma, \sigma^{-1}] \xrightarrow{\theta} \mathbb{C}[\sigma, \sigma^{-1}] \xrightarrow{\gamma} \ldots$$

where $\eta = 1_{\mathcal{K}}$, while θ and γ are defined on generators by

$$\theta(1) = 0, \quad \theta(\sigma^i) = \sigma^i, \;\; i \neq 0,$$

$$\gamma(1) = 1, \quad \gamma(\sigma^j) = 0, \;\; j \neq 0.$$

One then finds that $\mathrm{Cotor}^*_{\mathcal{K}}(\mathbb{C}, \mathcal{H}_1^{\otimes q}) = \mathcal{H}_1^{\otimes q}$ with the following coboundary:

$$\mathcal{H}_1^{\otimes q} \xrightarrow{\tilde{\theta}} \mathcal{H}_1^{\otimes q} \xrightarrow{\tilde{\gamma}} \mathcal{H}_1^{\otimes q} \xrightarrow{\tilde{\theta}} \mathcal{H}_1^{\otimes q} \xrightarrow{\tilde{\gamma}} \ldots,$$

where $\tilde{\theta}$ and $\tilde{\gamma}$ are defined by

$$\tilde{\theta}(\tilde{h}) = 0, \text{ if } |\tilde{h}| = -k, \qquad \tilde{\theta}(\tilde{h}) = \tilde{h}, \text{ if } |\tilde{h}| \neq -k,$$

$$\tilde{\gamma}(\tilde{h}) = \tilde{h}, \text{ if } |\tilde{h}| = -k \quad \tilde{\gamma}(\tilde{h}) = 0, \text{ if } |\tilde{h}| \neq -k.$$

So the resulting cohomology is as follows:

$$E_1^{p,q} = 0,\ p > 0, \quad E_1^{0,q} = \mathcal{H}^q_{1\natural}[-k] := \{\tilde{h} \in \mathcal{H}_1^{\otimes q} \mid |\tilde{h}| = -k\},$$

or schematically

$$\begin{array}{cccc}
 & \vdots & \vdots & \\
 & \mathcal{H}_1^{\otimes 2}[-k] & 0 & \cdots \\
E_1 & \mathcal{H}_1[-k] & 0 & \cdots \\
 & 0 & 0 & \cdots
\end{array}$$

To compute the E_2 term of this spectral sequence one notes that the vertical operators are exactly those of the cocyclic submodule $\mathcal{H}_{1\natural}[-k]$,

$$\begin{array}{cccc}
 & \vdots & \vdots & \\
 & {\scriptstyle b_{\mathcal{H}_1}}\uparrow & & \\
 & \mathcal{H}_1^{\otimes 2}[-k] & 0 & \cdots \\
 & {\scriptstyle b_{\mathcal{H}_1}}\uparrow & & \\
E_2 & \mathcal{H}_1[-k] & 0 & \cdots \\
 & {\scriptstyle b_{\mathcal{H}_1}}\uparrow & & \\
 & \mathbb{C}[-k] & 0 & \cdots
\end{array}$$

Therefore

$$E_2^{p,q} = 0,\ p \geq 0, \qquad E_2^{0,q} = H^q_{(\delta,1)}(\mathcal{H}_{1\natural}[-k]),$$

where the lower subscript for the cohomology groups stands for the coefficients.

Thus, the Hochschild cohomology of $\mathcal{H}_1^\dagger$ with coefficients in (δ, σ^k) is

$$H^n_{(\delta,\sigma^k)}(\mathcal{H}_1^\dagger) = H^n_{(\delta,1)}(\mathcal{H}_{1\natural}[-k]).$$

Since the vertical operators coincide with those belonging to the cocyclic structure for $(\mathcal{H}_1; \delta, 1)$ we conclude that

$$HP^*_{(\delta,\sigma^k)}(\mathcal{H}_1^\dagger) = HP^*_{(\delta,1)}(\mathcal{H}_{1\natural}[-k]) \tag{4.7}$$

Remark that all the preceding arguments work as well when the $\mathcal{H}_1^\dagger$ is replaced by any of the 'finite' cyclic covers $\mathcal{H}_1^{\dagger|N}$, with $N > 1$.

Theorem 4.8. 1^0. *Let $N > 1$. Of all the MPIs (δ, σ^k), $k \in \mathbb{Z}$, only (δ, σ^{-1}) yields nontrivial periodic cyclic cohomology:*

$$HP^*_{(\delta,\sigma^k)}(\mathcal{H}_1^{\dagger|N}) = 0 \quad \text{if} \quad k \neq -1, \quad \text{and} \quad HP^*_{(\delta,\sigma^{-1})}(\mathcal{H}_1^{\dagger|N}) \cong HP^*_{(\delta,1)}(\mathcal{H}_1).$$

2^0. *The periodic cyclic cohomology of $\mathcal{H}_1^{\dagger|N}$, $N > 1$, with coefficients in (δ, σ^{-1}) is generated by $[TF^\dagger]$ in the even degree and by $[\delta_1^\dagger]$ in the odd degree, where*

$$TF^\dagger = \sigma^{-1}X \otimes \sigma^{-1}Y - Y \otimes \sigma^{-1}X - \sigma^{-1}\delta_1 Y \otimes \sigma^{-1}Y, \quad \delta_1^\dagger = -\sigma^{-1}\delta_1.$$

Proof. For the first claim, it suffices to apply Corollary 3.10 to the right hand side of (4.7).
To prove the second, we start from the known generators $[\delta_1]$ and $[TF] = [X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y]$ for the periodic cyclic cohomology of $\mathcal{H}_1$ and go through the two moves required by the above routine to convert them into periodic cyclic cohomology classes for $\mathcal{H}_1^\dagger$. The first is to transfer back these classes to the total complex via Alexander-Whitney map (2.30), and the second amounts to applying Ψ^{-1}.
Concerning the first move, we note that for $\tilde{h} \in E^{0,q}$ one has

$$\overrightarrow{\partial}_2 \overrightarrow{\partial}_1(\tilde{h}) = \sigma^{(|\tilde{h}|-1)} \otimes \sigma^{(|\tilde{h}|-1)} \otimes \tilde{h}\,;$$

indeed,

$$\overrightarrow{\partial}_2 \overrightarrow{\partial}_1(h^1 \otimes h^2) = \overrightarrow{\partial}_2(\sigma^{-1} h^1{}_{<-1>} h^2{}_{<-1>} \otimes h^1{}_{<0>} \otimes h^2{}_{<0>}) =$$
$$\overrightarrow{\partial}_2(\sigma^{|h^1|+|h^2|-1} \otimes h^1 \otimes h^2) = \sigma^{|h^1|+|h^2|-1} \otimes \sigma^{-1} h^1{}_{<-1>} h^2{}_{<-1>} \otimes h^1{}_{<0>} \otimes h^2{}_{<0>}$$
$$= \sigma^{|h^1|+|h^2|-1} \otimes \sigma^{|h^1|+|h^2|-1} \otimes h^1 \otimes h^2.$$

Applying this to TF one obtains

$$AW(TF) = 1 \otimes 1 \otimes X \otimes Y - 1 \otimes 1 \otimes Y \otimes X - 1 \otimes 1 \otimes \delta_1 Y \otimes Y$$

In the second move, we get

$$\begin{aligned}
&\Psi^{-1}(1 \otimes 1 \otimes X \otimes Y) = X \rtimes \sigma^{-1} \otimes Y \rtimes \sigma^{-1} = \sigma^{-1} X \otimes \sigma^{-1} Y, \\
&\Psi^{-1}(1 \otimes 1 \otimes Y \otimes X) = Y \rtimes 1 \otimes X \rtimes \sigma^{-1} = Y \otimes \sigma^{-1} X, \\
&\Psi^{-1}(1 \otimes 1 \otimes \delta_1 Y \otimes Y) = \delta_1 Y \rtimes \sigma^{-1} \otimes Y \rtimes \sigma^{-1} = \sigma^{-1} \delta_1 Y \otimes \sigma^{-1} Y,
\end{aligned}$$

which implies that

$$\Psi^{-1} \circ AW(TF) = \sigma^{-1} X \otimes \sigma^{-1} Y - Y \otimes \sigma^{-1} X - \ \sigma^{-1} \delta_1 Y \otimes \sigma^{-1} Y.$$

Repeating the procedure for the class δ_1,one obtains

$$AW(\delta_1) = -1 \otimes \delta_1 \quad \text{and} \quad \Psi^{-1}(1 \otimes \delta_1) = \delta_1 \rtimes \sigma^{-1} = \sigma^{-1} \delta_1 \,,$$

and therefore

$$\Psi^{-1} \circ AW(\delta_1) = -\sigma^{-1} \delta_1 \,.$$

□

4.3 Hopf cyclic cohomology of $\mathcal{H}_{CK}$ and $\mathcal{H}_{CK}^{\dagger}$

In this subsection we apply the same method as in the previous two in order to calculate the Hopf cyclic cohomology of the extended Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}$.

We begin by recalling from [5] the basic ingredients from which the Hopf algebra $\mathcal{H}_{CK}$ is constructed.
A *rooted tree* T is a finite, connected and simply connected 1-dimensional simpicial complex with a base point (root) $* \in \Delta^0 = \{\text{set of vertices of } T\}$. The set of rooted trees, up to isomorphism, is denoted by Σ. It is graded and by weight, which is defined as the integer

$$|T| = Card \Delta^{(0)}(T) = \#\,\text{of vertices of}\ T\,.$$

A *cut* of a rooted tree T is a subset $c \subset \Delta^{(1)}(T)$ of the set of edges of T. A cut is called *simple* if for any $x \in \Delta^{(0)}(T)$ the path $(*, x)$ only contains

one element of c. To any simple cut of a rooted tree T one assigns a trunk denoted by $R_c(T)$ and a collection of cut branches denoted by $P_c(T)$.
Let $\mathcal{H}_{rt}$ be the free commutative algebra generated by the symbols

$$\{\delta_T\,;\quad T\in\Sigma\}\,.$$

One defines a coproduct on $\mathcal{H}_{rt}$ by extending to an algebra homomorphism the map defined on the generators of $\mathcal{H}_{rt}$ by

$$\Delta(\delta_T)=\delta_T\otimes 1+\sum_{\text{simple cuts}}\left(\prod_{P_c(T)}\delta_{T_i}\right)\otimes\delta_{R_c(T)}.$$

We let $\mathfrak{g}$, X, and Y be the same as in (4.1), and let $\mathcal{U}:=\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. It acts from right on $\mathcal{H}_{rt}$ as follows:

$$\delta_T\triangleleft Y=-|T|\delta_T,$$
$$\delta_T\triangleleft X=-N\delta_T:=-\sum\delta_{T'}\,,$$

where the trees T' are obtained by adding one vertex and one edge to T in all possible ways without changing the base point. One then extends the action of $\mathcal{U}$ on $\mathcal{H}_{rt}$, so that $\mathcal{H}_{rt}$ becomes a $\mathcal{U}$-module algebra.
On the other hand, as in the case of $\mathcal{H}_1$, one defines a left coaction of $\mathcal{H}_{rt}$ on $\mathcal{U}$, $\rho:\mathcal{U}\to\mathcal{H}_{rt}\otimes\mathcal{U}$, by setting

$$\rho(X)=1\otimes X+\delta_*\otimes Y\,,$$
$$\rho(Y)=1\otimes Y\,,$$

and then extending the map ρ to $\mathcal{U}$ via the rule (4.5).
Equipped with this coaction $\mathcal{U}$ is a left $\mathcal{H}_{rt}$-comodule algebra. Moreover the action and coaction thus defined fulfill the axioms of a matched pair [22, p. 232].
One then forms a bicrossed Hopf algebra $\mathcal{H}_{CK}\equiv\mathcal{U}\blacktriangleright\!\!\triangleleft\,\mathcal{H}_{rt}$, whose algebra structure is the crossed product algebra $\mathcal{U}\ltimes\mathcal{H}_{rt}$, and coalgebra structure is $\mathcal{U}\rtimes\mathcal{H}_{rt}$, with antipode

$$S(u\rtimes h)=(1\rtimes S_{\mathcal{H}_{rt}}(u_{<-1>}h))(S_{\mathcal{U}}(h_{<0>})\rtimes 1).$$

We identify $X \rtimes 1$, $Y \rtimes 1$ and $1 \rtimes \delta_T$ with X, Y, and δ_T respectively. One can easily see that, with the character δ defined as before,

$$\delta(X) = 0, \quad \delta(Y) = 1, \quad \delta(\delta_T) = 0\,,$$

$(\delta, 1)$ is a MPI for $\mathcal{H}_{CK}$.
So the Hopf cyclic cohomology of $\mathcal{H}_{CK}$ with coefficients in $(\delta, 1)$ is well defined, and to compute it we only need to repeat the method of the preceding subsections.

Let $\mathcal{H}_{CK\natural}$ be the cocyclic module associated to $(\mathcal{H}_{rt}; \delta, 1)$. Since the operators defining the cyclic structure are weight preserving, the cocyclic module is graded decomposes into eigenmodules

$$\mathcal{H}_{CK\natural} = \bigoplus_{k \geq 0} \mathcal{H}_{CK\natural}[k]\,,$$

where $\mathcal{H}_{CK\natural}[k]$ consists of all elements in $\mathcal{H}_{CK\natural}[k]$ of weight k.
Furthermore, as Y is a primitive element of $\mathcal{H}_{CK}$ and since ad_Y implements the grading, using the the Cartan homotopy formula 3.10, one obtains that

$$HP^*(\mathcal{H}_{CK\natural}[1]) = HP^*_{(\delta,1)}(\mathcal{H}_{CK})\,.$$

We first compute the Hochschild cohomology of left hand side. To this end we again use the bicosimplicial module

$$\mathfrak{C}^{p,q} = \mathcal{H}_{rt}^{\otimes p} \otimes \mathcal{U}^{\otimes q},$$

with the cosimplicial structure defined by (2.13)–(2.19).
The following defines a filtration on the total complex of the bicomplex $(\mathfrak{C}, \overrightarrow{b}, \uparrow b)$, where the horizontal boundary $\overrightarrow{b}$ and and the vertical boundary $\uparrow b$ are defined in (2.26):

$$F_i = \bigoplus_{p \geq i} \bigoplus_{q \geq 0} \mathcal{H}_{rt}^{\otimes p} \otimes \mathcal{U}^{\otimes q}$$

and the associated spectral sequence E_r converges to the Hochschild cohomology of the total complex $\mathrm{Tot}(\mathfrak{C})$.
The spectral sequence E_r inherits the grading of $\mathcal{H}_{CK}$, since all cosimplicial maps, as well as Ψ and Ψ^{-1} (defined in §2.2), are weight preserving. As

before, in view of the Cartan homotopy formula, it suffices to focus on its weight 1 component $E_r[1]$.
Using again the fact that the antisymmetrisation map induces an isomorphism between coalgebra Hochschild cohomology of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ and its exterior algebra $\wedge^*\mathfrak{g}$, one obtains the term E_1 of the spectral sequence as follows.

Proposition 4.9. *$E_1^{p,0} = \mathcal{H}_{rt}^{\otimes p}$, $E_1^{p,1} = (\mathcal{H}_{rt}^{\otimes p} \otimes X) \otimes (\mathcal{H}_{rt}^{\otimes p} \otimes Y)$, $E_1^{p,q} = 0$ for $q \geq 3$ and $E_1^{p,2} = \mathcal{H}_{rt}^{\otimes p} \otimes X \wedge Y$.*

Since up to a constant factor δ_* is the only element of weight 1 in $\mathcal{H}_{rt}$, and taking into consideration that we are always working in the normalized complex, we obtain the following picture for $E_2[1]$:

$$\begin{array}{ccccccc}
\mathbb{C}X \wedge Y & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\\
\mathbb{C}[X] & \xrightarrow{\vec{b}} & \mathbb{C}\delta_* \otimes \mathbb{C}Y & \longrightarrow & 0 & \longrightarrow & \cdots \\
\\
0 & \longrightarrow & \mathbb{C}\delta_* & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Because $\vec{b}(X) = \delta_* \otimes Y$, one sees that only $[\delta_*]$, and $X \wedge Y$ survive in $E_2[1]$. By the same arguments as in the proof of Proposition 4.3, we obtain that the Hochschild cohomology in weight 1 is generated by the classes

$$[\delta_*] \quad \text{and} \quad [X \otimes Y - Y \otimes X - Y\delta_* \otimes Y] \,.$$

Finally, one computes the periodic cyclic cohomology of $\mathcal{H}_{CK}$ exactly as in the case of $\mathcal{H}_1$. In particular, the proof of the following result is identical to that of Theorem 4.4.

Theorem 4.10. *The periodic cyclic cohomology of the Hopf algebra $\mathcal{H}_{CK}$ with respect to the MPI $(\delta, 1)$ is the following:*

$$HP^1_{(\delta,1)}(\mathcal{H}_{CK}) = \mathbb{C} \cdot [\delta_*] \;\; \textit{and} \;\; HP^0_{(\delta,1)}(\mathcal{H}_{CK}) = \mathbb{C} \cdot [X \otimes Y - Y \otimes X - \delta_* Y \otimes Y].$$

We end with the observation that the Hopf algebra $\mathcal{H}_{CK}$, admits its own 'cyclic coverings' too. Indeed, as it was done for $\mathcal{H}_1$ in (4.6), one can endow it with a similar coaction $\rho : \mathcal{H}_{CK} \to \mathcal{K} \otimes \mathcal{H}_{CK}$,

$$\rho(h) = \sigma^{|h|} \otimes h \,.$$

One checks that this coaction turns $\mathcal{H}_{CK}$ into a $\mathcal{K}$-comodule Hopf algebra, as in the proof of Lemma 4.6. We can thus form the cocrossed product Hopf algebra

$$\mathcal{H}_{CK}^{\dagger} := \mathcal{H}_{CK} \rtimes \mathcal{K}.$$

The periodic Hopf cyclic cohomology of $\mathcal{H}_{CK}^{\dagger}$ as well as of the finite cyclic covers $\mathcal{H}_{CK}^{\dagger|N}$, $N > 1$, with coefficients in the MPIs (δ, σ^k), $k \in \mathbb{Z}$, can be computed in exactly the same way as for $\mathcal{H}_1^{\dagger}$ (see Theorem 4.8). We record below the analogous results.

Theorem 4.11. 1^0. *Let $N > 1$. Of all the MPIs (δ, σ^k), $k \in \mathbb{Z}$, only (δ, σ^{-1}) yields nontrivial periodic cyclic cohomology:*

$$HP^*_{(\delta,\sigma^k)}(\mathcal{H}_{CK}^{\dagger|N}) = 0 \quad \text{if} \quad k \neq -1, \quad \text{and} \quad HP^*_{(\delta,\sigma^{-1})}(\mathcal{H}_{CK}^{\dagger|N}) \cong HP^*_{(\delta,1)}(\mathcal{H}_{CK}).$$

2^0. *The periodic cyclic cohomology of $\mathcal{H}_{CK}^{\dagger|N}$ with coefficients in (δ, σ^{-1}) is generated by $[TF^{\dagger}]$ in the even degree and by $[\delta_*^{\dagger}]$ in the odd degree, where*

$$TF^{\dagger} = \sigma^{-1}X \otimes \sigma^{-1}Y - Y \otimes \sigma^{-1}X - \sigma^{-1}\delta_* Y \otimes \sigma^{-1}Y, \quad \delta_*^{\dagger} = -\sigma^{-1}\delta_*.$$

References

[1] Akbarpour, R. and Khalkhali, M., Hopf algebra equivariant cyclic homology and cyclic homology of crossed product algebras, J. Reine Angew. Math., **559** (2003), 137-152.

[2] Connes, A., Spectral sequence and homology of currents for operator algebras, *Math. Forschungsinst. Oberwolfach Tagungsber* 41/81; *Funktionalanalysis und C^*-Algebren*, 27-9/3-10, 1981.

[3] Connes, A., Cohomologie cyclique et foncteur Ext^n, C.R. Acad. Sci. Paris Ser. I Math., **296** (1983), 953-958.

[4] Connes, A., Noncommutative differential geometry, Publ. Math. Inst. Hautes Etudes Sci., **62** (1985), 257–360.

[5] Connes, A. and Kreimer, D., Hopf algebras, renormalization and noncommutative geometry, in *Quantum field theory: perspective and prospective* (Les Houches, 1998), 59–108, NATO Sci. Ser. C Math. Phys. Sci., **530**, Kluwer Acad. Publ., Dordrecht, 1999.

[6] Connes, A. and Moscovici, H., The local index formula in noncommutative geometry, Geom. Funct. Anal., **5** (1995), 174-243.

[7] Connes, A. and Moscovici, H., Hopf algebras, cyclic cohomology and the transverse index theorem, Commun. Math. Phys. **198** (1998), 199-246.

[8] Connes, A. and Moscovici, H., Cyclic cohomology and Hopf algebras. Lett. Math. Phys., **48** (1999), no. 1, 97–108.

[9] Connes, A. and Moscovici, H., Cyclic cohomology and Hopf algebra symmetry, Conference Moshé Flato 1999 (Dijon). Lett. Math. Phys., **52** (2000), no. 1, 1–28.

[10] Connes, A. and Moscovici, H., Modular Hecke algebras and their Hopf symmetry, Moscow Math. J., **4** (2004), 67–109.

[11] Connes, A. and Moscovici, H., Rankin-Cohen brackets and the Hopf algebra of transverse geometry, Moscow Math. J., **4** (2004), 111–130 (arXiv:math.QA/0304316).

[12] Connes, A. and Moscovici, H., Background independent geometry and Hopf cyclic cohomology, arXiv:math.QA/0505475.

[13] Crainic, M., Cyclic cohomology of Hopf algebras, J. Pure Appl. Algebra, **166** (2002), no. 1-2, 29–66.

[14] Feĭgin, B. L. and Tsygan, B. L., Additive K-theory, in *K-theory, arithmetic and geometry* (Moscow, 1984–1986), 67–209, Lecture Notes in Math., **1289**, Springer, Berlin, 1987.

[15] Fuks, D. B., Cohomology of infinite-dimensional Lie algebras, Consultants Bureau, New York, 1986.

[16] Getzler, E. and Jones, J. D. S., The cyclic homology of crossed product algebras, J. Reine Angew. Math., **445** (1993), 163–174.

[17] Hajac, P. M., Khalkhali, M., Rangipour, B. and Sommerhäuser Y., Stable anti-Yetter-Drinfeld modules, C. R. Math. Acad. Sci. Paris, **338** (2004), no. 8, 587–590.

[18] Hajac, P. M., Khalkhali, M., Rangipour, B. and Sommerhäuser Y., Hopf-cyclic homology and cohomology with coefficients, C. R. Math. Acad. Sci. Paris, **338** (2004), no. 9, 667–672.

[19] Kassel, C., **Quantum groups**, Graduate Texts in Mathematics, **155**, Springer-Verlag, New York, 1995.

[20] Khalkhali, M. and Rangipour B., Introduction to Hopf-Cyclic Cohomology, in *Noncommutative Geometry and Number Theory*, pp.155–178, Aspects of Mathematics **E 37**, Vieweg, 2006 (arXiv: math.QA/0503244).

[21] Loday, J.-L., **Cyclic homology**, Fundamental Principles of Mathematical Sciences, **301**, Springer-Verlag, Berlin, 1992.

[22] Majid, S., **Foundations of quantum group theory**, Cambridge University Press, Cambridge, 1995.